Categorical probability in the quantum realm

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About 90% of this talk is on https://arxiv.org/abs/2001.08375 and the other 10% of this talk is based on joint work with Benjamin Russo at SUNY Farmingdale in New York and is available at https://arxiv.org/abs/1907.09689 and https://arxiv.org/abs/2005.03886.

fdC*-AlgU$^{\between}$ as a category

The objects of **fdC*-AlgU**$^{\between}$ are finite-dimensional unital C^*-algebras, which are all of the form (up to isomorphism)

$$\mathcal{A} = \bigoplus_{x \in X} \mathcal{M}_{m_x},$$

where X is a finite set and $\mathcal{M}_{m_x}$ is the unital $*$-algebra of all $m_x \times m_x$ matrices equipped with the operator norm and conjugate transpose as the involution $*$.

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A morphism from $\mathcal{B}$ to $\mathcal{A}$ is either a linear unital map or a *conjugate* linear unital map $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$. The latter means $F(\lambda b) = \overline{\lambda} F(b)$ for all $\lambda \in \mathbb{C}$ and $b \in \mathcal{B}$, where $\overline{\lambda}$ is the conjugate transpose of λ.

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\begin{aligned}
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fdC*-AlgU$^{\between}$ as a $\mathbb{Z}_2$-graded $\otimes$-category

Thus, **fdC*-AlgU**$^{\between}$ is not a monoidal category with the usual tensor product. Instead, we can only take the tensor product of "even" morphisms (linear maps) and "odd" morphisms (conjugate linear maps).

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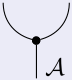
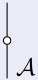
This can be made precise with the notion of G-graded monoidal categories of Fröhlich and Wall.

Quantum Markov categories

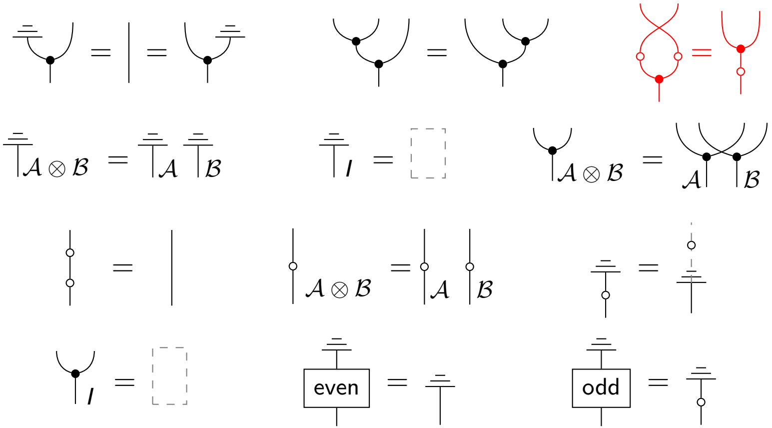
Definition

A **quantum Markov category** (QMC) is a $\mathbb{Z}_2$-monoidal category $\mathcal{M}$ together with a family of morphisms **copy** $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \rightsquigarrow \mathcal{A}$, **discard** $!_{\mathcal{A}} : I \rightsquigarrow \mathcal{A}$, and **involve** $*_{\mathcal{A}} : \mathcal{A} \rightsquigarrow \mathcal{A}$, all depicted in string diagram notation as

$\mu_{\mathcal{A}} \equiv$  , $!_{\mathcal{A}} \equiv$  , and $*_{\mathcal{A}} \equiv$  ,

for all objects $\mathcal{A}$ in $\mathcal{M}$. These morphisms are required to satisfy several conditions.

QMC String diagrams



QMC String diagrams for **fdC*-AlgU**$^{\between}$

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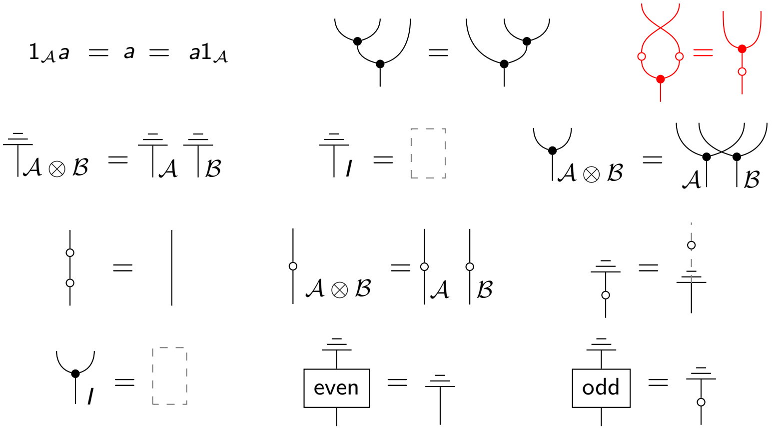

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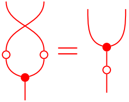

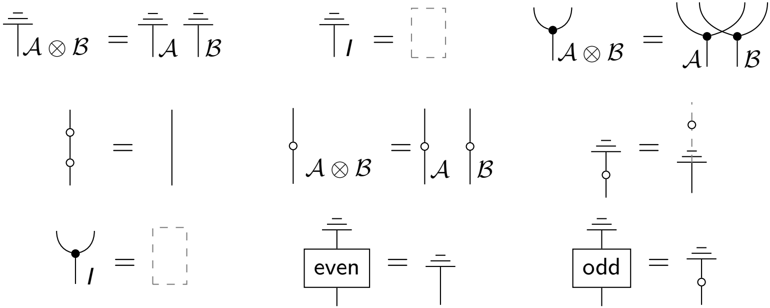

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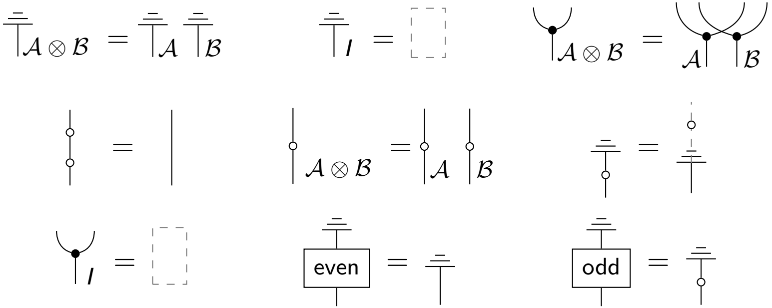

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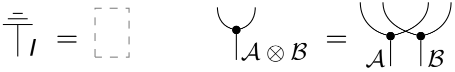

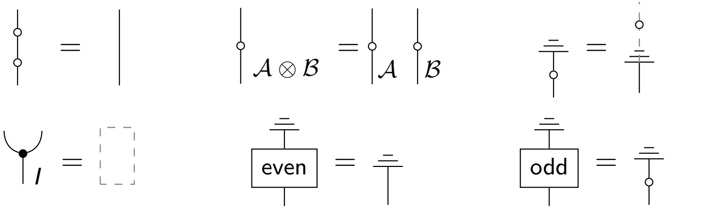

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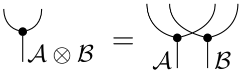

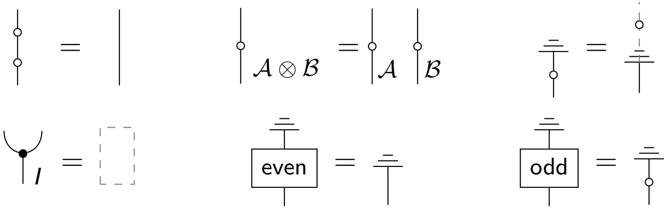

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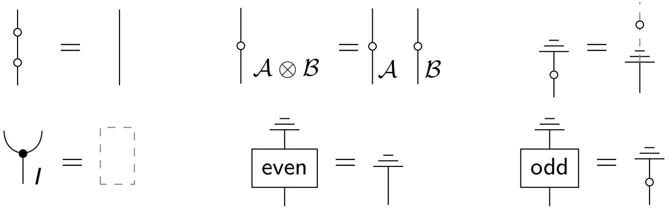

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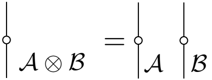

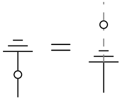

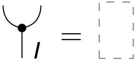

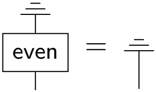

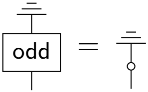

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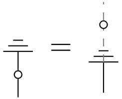

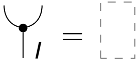

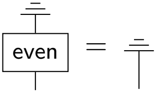

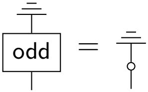

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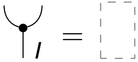

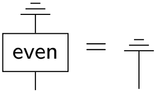

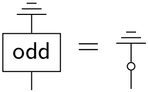

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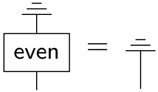

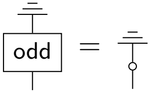

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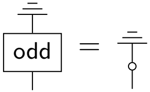

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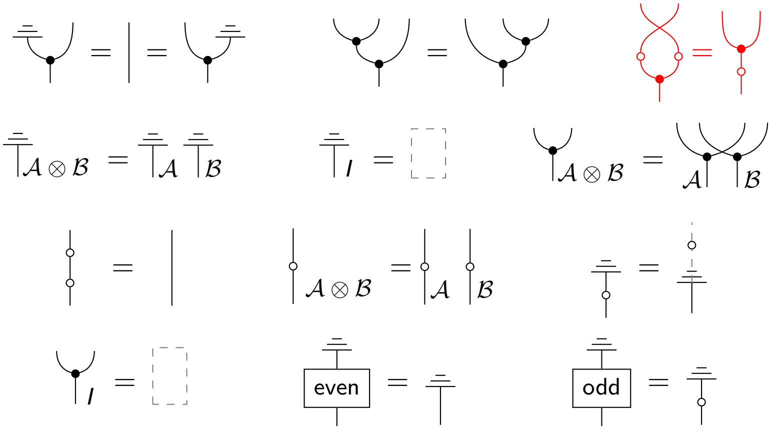

(Classical) Markov categories

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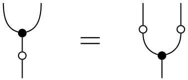

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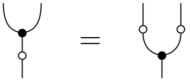

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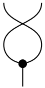

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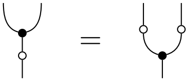

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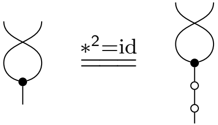

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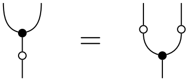

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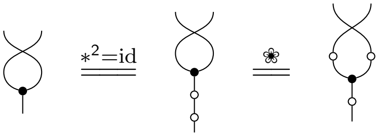

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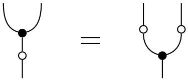

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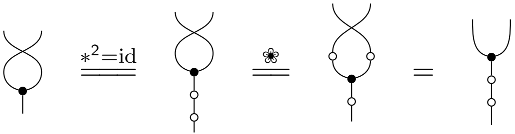

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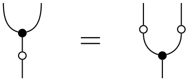

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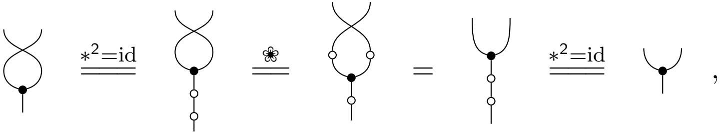

which reproduces the usual definition of a Markov category.

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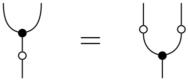

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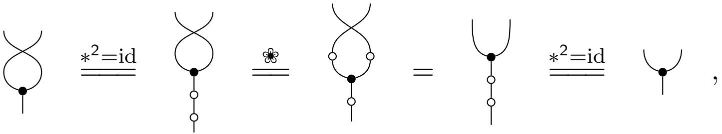

which reproduces the usual definition of a Markov category. In the case of **fdC*-AlgU**$^{\between}$, the subcategory of *commutative* finite-dimensional C^*-algebras and positive unital maps (defined shortly) is equivalent to **FinStoch**$^{\text{op}}$, the category of finite sets and stochastic maps.

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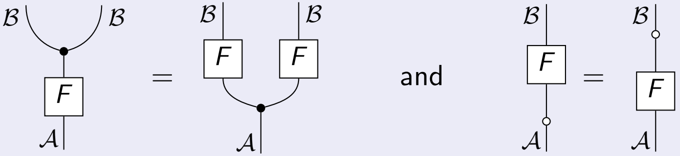

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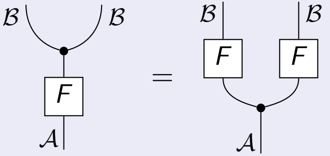

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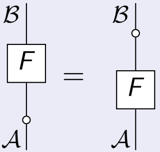

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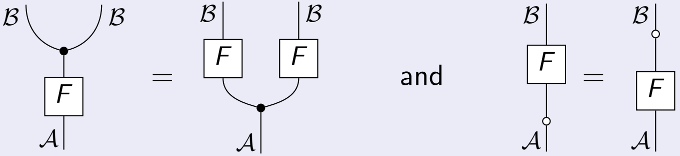

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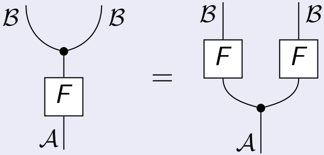

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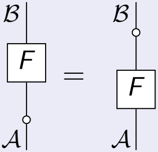

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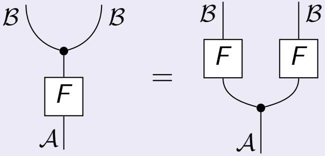

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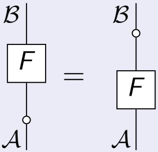

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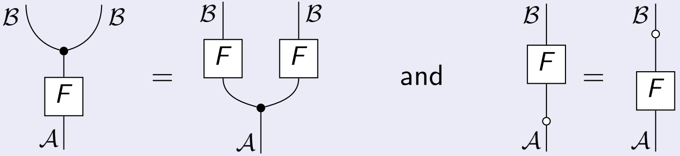

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$$\textbf{fdC*-AlgDU}\subseteq\textbf{fdC*-AlgCPU}\subseteq\textbf{fdC*-AlgSPU}\subseteq\textbf{fdC*-AlgPU}.$$

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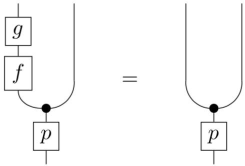
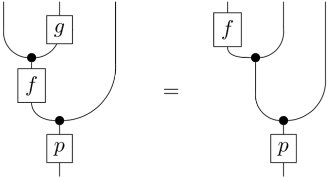


holds for any morphisms as indicated, then also



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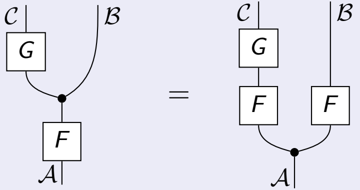
Definition

Let $\mathcal{M}$ be a quantum Markov category. A subcategory $\mathcal{P} \subseteq \mathcal{M}_{\text{even}}$ is said to be **S-positive** in $\mathcal{M}$ iff for every pair of composable morphisms $\mathcal{C} \overset{G}{\rightsquigarrow} \mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ in $\mathcal{P}$ such that $F \circ G$ is deterministic, then



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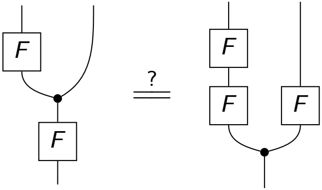

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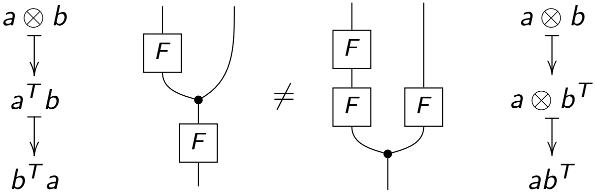

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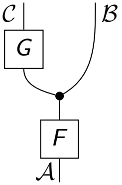
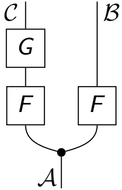


$$F\big(G(c)b\big) \quad = \quad F\big(G(c)\big)F(b)$$



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Let $\mathcal{C} \overset{G}{\rightsquigarrow} \mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ be a pair of composable SPU maps of C^*-algebras such that the composite $F \circ G$ is a *-homomorphism. Then,

$$\begin{aligned} F\big(G(c)^*G(c)\big) &\le F\big(G(c^*c)\big) && \text{by S-positivity of } G \\ &= F\big(G(c)\big)^* F\big(G(c)\big) && \text{since } F \circ G \text{ is deterministic} \\ &\le F\big(G(c)^*G(c)\big) && \text{by S-positivity of } F \end{aligned}$$

holds for all $c \in \mathcal{C}$.

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By the Multiplicative Theorem, this implies

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Since F and G are $*$-preserving (natural with respect to $*$) and $*$ is an involution, this reproduces the required condition.

fdC*-AlgCPU is an S-positive subcategory of **fdC*-AlgU**$^{\between}$

fdC*-AlgCPU is also an S-positive subcategory of **fdC*-AlgU**$^{\between}$ (in fact, the subcategory of n-positive unital maps is as well for all $n \geq 2$).

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Question

Is **fdC*-AlgSPU** *the largest S-positive subcategory of* **fdC*-AlgU**$^{\between}$*?*

fdC*-AlgCPU is an S-positive subcategory of **fdC*-AlgU**$^{\between}$

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Question

Is **fdC*-AlgSPU** *the largest S-positive subcategory of* **fdC*-AlgU**$^{\between}$*?*

Question

Are there diagrammatic axioms that characterize the subcategory **fdC*-AlgPU** *of positive unital maps inside* **fdC*-AlgU**$^{\between}$*?*

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Question

Are there diagrammatic axioms that characterize the subcategory **fdC*-AlgPU** *of positive unital maps inside* **fdC*-AlgU**$^{\between}$*?*

Question

Which subcategories of **fdC*-AlgU**$^{\between}$ *obey Fritz' first (before v. IV) notion of positive subcategory?*

fdC*-AlgCPU is an S-positive $\otimes$-subcat of **fdC*-AlgU**$^{\between}$

Since CP maps are S-positive, **fdC*-AlgCPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$.

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Since CP maps are S-positive, **fdC*-AlgCPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$. Unlike **fdC*-AlgSPU**, however, **fdC*-AlgCPU** is closed under the tensor product. Thus, **fdC*-AlgCPU** is an S-positive *monoidal* subcategory of **fdC*-AlgU**$^{\between}$.

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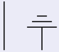
Question

Is **fdC*-AlgCPU** *the largest S-positive monoidal subcategory of* **fdC*-AlgU**$^{\between}$*?*

A no-cloning theorem for S-positive subcategories

Theorem (No broadcasting for S-positive subcategories)

Let $\mathcal{P}$ be an S-positive subcategory of a quantum Markov category $\mathcal{M}$ containing the morphisms ,  , *and*  *for each object in $\mathcal{P}$.*

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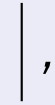
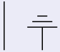
Let $\mathcal{P}$ be an S-positive subcategory of a quantum Markov category $\mathcal{M}$ containing the morphisms ,  , *and*  *for each object in $\mathcal{P}$. In addition, suppose that $\mathcal{P}$ contains a morphism*  *satisfying*

for every object in $\mathcal{P}$.

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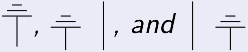
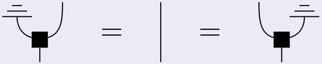
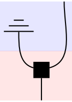
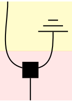
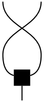
Let $\mathcal{P}$ be an S-positive subcategory of a quantum Markov category $\mathcal{M}$ containing the morphisms  *for each object in $\mathcal{P}$. In addition, suppose that $\mathcal{P}$ contains a morphism*  *satisfying*



for every object in $\mathcal{P}$. Then  *is commutative and in fact equals duplication for every object of $\mathcal{P}$.*

Proof of no-cloning for S-positive subcategories

Since  $=$ $=$ , which is deterministic, S-positivity gives



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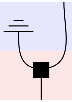
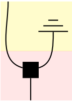
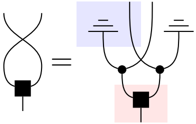
Since  = = , which is deterministic, S-positivity gives



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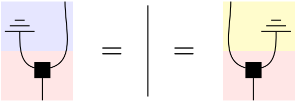
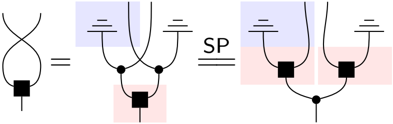
Since , which is deterministic, S-positivity gives



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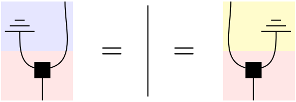
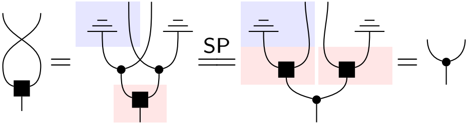
Since , which is deterministic, S-positivity gives



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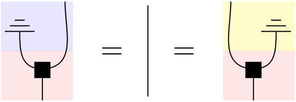
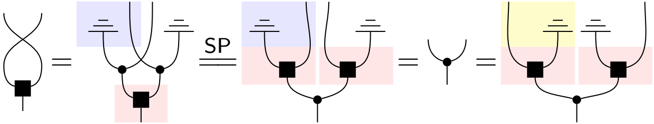
Since  , which is deterministic, S-positivity gives



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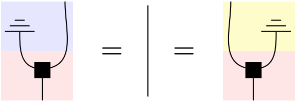
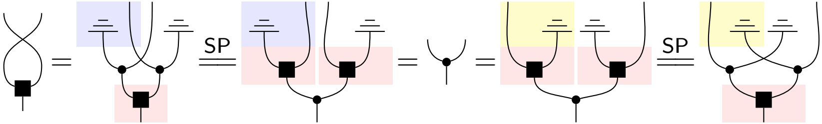
Since  , which is deterministic, S-positivity gives



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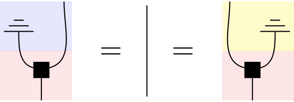
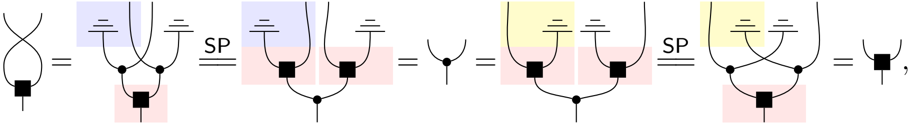
Since , which is deterministic, S-positivity gives



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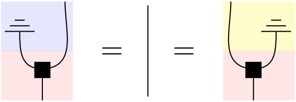
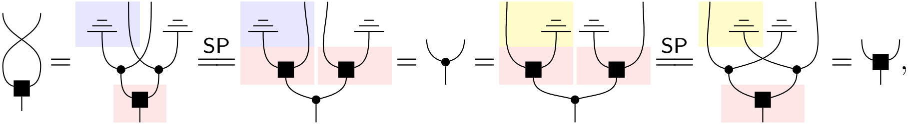
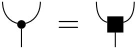
Since , which is deterministic, S-positivity gives



which reproduces the commuting axiom since .

Almost everywhere equivalence $F \underset{\omega}{=} G$

Theorem (P. 2001.08375 [quant-ph])

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, let $F, G : \mathcal{B} \rightsquigarrow \mathcal{A}$ be two linear maps, and let $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ be a state (a PU map). Then the following are equivalent.

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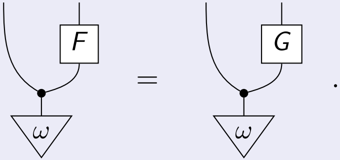

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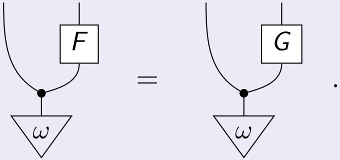

In this case, F is said to be ω-**a.e. equivalent to** G.

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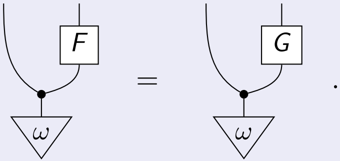
ii. 

In this case, F is said to be ω-**a.e. equivalent to** G. The first definition appears in 1907.09689 [quant-ph] and the second (for ordinary Markov categories) is due to Cho–Jacobs 1709.00322 [cs.AI].

Bayes' theorem

Theorem (Bayes' theorem)

Let X and Y be finite sets, let $\{\bullet\} \overset{p}{\rightsquigarrow} X$ be a probability measure, and let $X \overset{f}{\rightsquigarrow} Y$ be a stochastic map.

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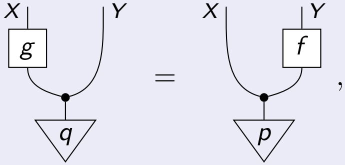

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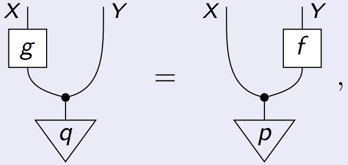

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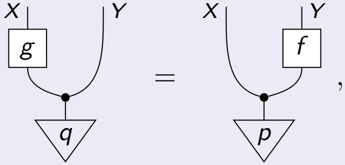

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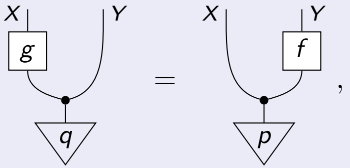

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You can watch my video explaining *why* I call this Bayes' theorem here.

Bayesian inverses

The previous theorem motivates the following definition.

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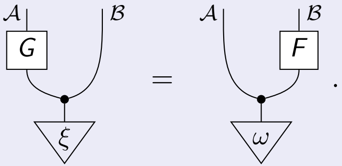
Definition

Let $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ be a CPU map, let $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ be a state, and set $\xi := \omega \circ F$. A **Bayesian inverse** of F is a CPU map $\mathcal{A} \overset{G}{\rightsquigarrow} \mathcal{B}$ such that



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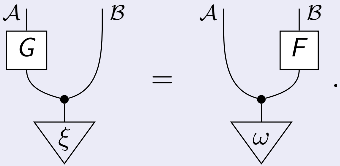

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Bayesian inverses

The previous theorem motivates the following definition.

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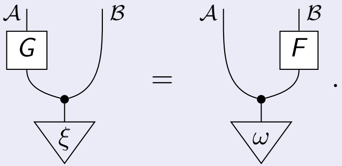

The existence of Bayesian inverses is not guaranteed for CPU maps between finite-dimensional C^*-algebras. A linear algebraic theorem characterizing its existence in **fdC*-AlgCPU** is given in 2005.03886 [quant-ph] (joint with Russo).

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Nevertheless, when they exist, Bayesian inverses satisfy many convenient properties.

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vi. *A tensor product of Bayesian inverses is a Bayesian inverse of the tensor product.*

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iii. *If F is deterministic and (F,ω) has a Bayesian inverse G, then G is a disintegration of (F,ω).*

All this and much more can be found in the following references.

- K. Cho and B. Jacobs "Disintegration and Bayesian Inversion via String Diagrams" 1709.00322 [cs.AI]
- T. Fritz "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics" 1908.07021 [math.ST]
- A. Parzygnat and B. Russo "Non-commutative disintegrations: existence and uniqueness in finite dimensions" 1907.09689 [quant-ph]
- A. Parzygnat "Inverses, disintegrations, and Bayesian inversion in quantum Markov categories" 2001.08375 [quant-ph]
- A. Parzygnat and B. Russo "A non-commutative Bayes' theorem" 2005.03886 [quant-ph]
- A. Parzygnat "Categorical probability theory" videos available at https://www.youtube.com/playlist?list=PLSx1kJDjrLRSKKHj4zetTZ45pVnGCRN80

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Thank you!