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Theorem Let $\mathcal{C}$ be a small category. Then there exists a co-complete category (a category that has all small colimits) $\underset{\sim}{\mathcal{C}}$ together with a functor $\mathcal{C} \xrightarrow{r} \underset{\sim}{\mathcal{C}}$ satisfying the following universal property: for any co-complete category $\mathcal{D}$ with a functor $\mathcal{C} \xrightarrow{\gamma} \mathcal{D}$ there exists a colimit-preserving functor $\underset{\sim}{\mathcal{C}} \xrightarrow{Re} \mathcal{D}$ so that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & \underset{\sim}{\mathcal{C}} \arrow[d, "Re"] \\ & \mathcal{D} \end{tikzcd}$$

commutes. Moreover, any two such functors $\underset{\sim}{\mathcal{C}} \to \mathcal{D}$ are canonically isomorphic.

Furthermore, Re has a right adjoint $\mathcal{D} \xrightarrow{Sing} \underset{\sim}{\mathcal{C}}$.

proof/construction Let $\underset{\sim}{\mathcal{C}} := Pre(\mathcal{C})$ the category of presheaves on $\mathcal{C}$ with values in Set and let $\mathcal{C} \xrightarrow{r} Pre(\mathcal{C})$ be the Yoneda embedding defined by

$$X \longmapsto \left(r(X) : Y \longmapsto \mathcal{C}(Y, X) \text{ on objects} \;\&\; \left(Y \xrightarrow{f} Z\right) \mapsto \left(\mathcal{C}(Z,X) \xrightarrow{-\circ f} \mathcal{C}(Y,X)\right) \text{ on morphisms} \right)$$

on objects

$$\text{and } \left(X \xrightarrow{\psi} X'\right) \longmapsto \left(r(\psi) : r(X) \Rightarrow r(X') \quad Y \longmapsto \left(\mathcal{C}(Y,X) \xrightarrow{\psi\circ -} \mathcal{C}(Y,X')\right)\right)$$

on morphisms.

The Yoneda Lemma states that this Yoneda embedding is full and faithful i.e. $\mathcal{C}(X, X') \cong Pre(\mathcal{C})(r(X), r(X'))$ canonically. Now let's prove that $\mathcal{C} \xrightarrow{r} Pre(\mathcal{C})$ satisfies the required universal property. Let $\mathcal{C} \xrightarrow{\gamma} \mathcal{D}$ be as above. To construct Re we will only use γ and the fact that objects of $Pre(\mathcal{C})$ are colimits of representables. More precisely, we recall the Density Theorem as a Lemma.

Lemma Let X be a presheaf of sets on $\mathcal{C}$. Let $\mathcal{C}\downarrow X$ be the category whose objects are pairs $(c, r(c) \longrightarrow X)$ in $\mathcal{C}$ & $Pre(\mathcal{C})$ respectively and whose morphisms are morphisms $c \to c'$ (or equivalently, by the Yoneda Lemma, $r(c) \to r(c')$) making

$$\begin{tikzcd} r(c) \arrow[rr] \arrow[dr] & & r(c') \arrow[dl] \\ & X & \end{tikzcd}$$

commute. Define a diagram in $Pre(\mathcal{C})$ by the functor

$$\mathcal{C}\downarrow X \xrightarrow{F} Pre(\mathcal{C}) \text{ that sends}$$

$$\left(c, \begin{array}{c} r(c) \\ \downarrow \\ X \end{array}\right) \longmapsto r(c) \text{ on objects and}$$

$$(c \to c') \longmapsto (r(c) \to r(c')) \text{ on morphisms}$$

Then $X \cong \operatorname{colim} F \equiv \operatorname*{colim}_{c\in\mathcal{C}, \mathcal{C}\downarrow X} r(c)$. We won't prove this.

Using this, we define Re to be

$$Re(X) := \operatorname*{colim}_{c\in\mathcal{C}, \mathcal{C}\downarrow X} \gamma(c) = \operatorname{colim} F_\gamma, \text{ where}$$

$\mathcal{C}\downarrow X \xrightarrow{F_\gamma} \mathcal{D}$ is defined by

$$\left(c, \begin{array}{c} r(c) \\ \downarrow \\ X \end{array}\right) \longmapsto \gamma(c) \text{ on objects and } (c \to c') \longmapsto (\gamma(c) \to \gamma(c')) \text{ on morphisms.}$$

This is technically not a functor but this issue can be taken care of. Because colimits are unique up to canonical isomorphism, any two such functors $Pre(\mathcal{C}) \overset{Re}{\underset{Re'}{\rightrightarrows}} \mathcal{D}$ that agree on $\mathcal{C}$ via γ, agree up to such a natural isomorphism. Now, define $\mathcal{D} \xrightarrow{Sing} Pre(\mathcal{C})$ to be the assignment

$$d \longmapsto \left(\begin{array}{l} \mathrm{Sing}(d) : c \longmapsto \mathcal{D}(\gamma c, d) \quad \text{on objects} \\ (c \xrightarrow{f} c') \longmapsto \left(\mathcal{D}(\gamma c', d) \xrightarrow{-\circ \gamma f} \mathcal{D}(\gamma c, d)\right) \\ \text{on morphisms} \end{array} \right)$$

on objects and

$$(d \xrightarrow{\psi} d') \longmapsto \left(\begin{array}{l} \mathrm{Sing}(\psi) : \mathrm{Sing}(d) \Rightarrow \mathrm{Sing}(d') \\ c \longmapsto \left(\mathcal{D}(\gamma c, d) \xrightarrow{\psi \circ -} \mathcal{D}(\gamma c, d')\right) \end{array} \right)$$

on morphisms. To check that Sing is a right adjoint of Re we need to show $\mathcal{D}(\mathrm{Re}(X), Y) \cong \hat{\mathcal{C}}(X, \mathrm{Sing}(Y))$ is an isomorphism natural in X and Y. We'll construct the map from right to left. Let $X \xrightarrow{\psi} \mathrm{Sing}(Y)$ be a natural transformation. So to every c we get a function $X(c) \xrightarrow{\psi_c} \mathcal{D}(\gamma c, Y)$ satisfying naturality for all $c \xrightarrow{f} c'$. Now, a morphism $\mathrm{Re}(X) = \operatorname*{colim}_{c \in \mathcal{C},\, \mathcal{C}\downarrow X} \gamma(c) \longrightarrow Y$ is specified by a collection of compatible morphisms $\gamma(c) \longrightarrow Y$ by the universal property of the colimit. The natural transformation $X \xrightarrow{\psi} \mathrm{Sing}(Y)$ defines such a cocone

$$\begin{tikzcd} \mathcal{C}\downarrow X \arrow[r, bend left, "F_\gamma"] \arrow[r, bend right, "\nabla_Y"'] \arrow[r, phantom, "\Downarrow \Psi"] & \mathcal{D} \end{tikzcd}$$

sending $\left(c, \begin{array}{c} r(c) \\ \downarrow \phi \\ X \end{array}\right)$ to a morphism $\gamma(c) \xrightarrow{\Psi_c} Y$. To see what this morphism should be look at the image of 1_c under $r(c)(c) = \mathcal{C}(c,c) \xrightarrow{\phi_c} X(c)$ and post compose with ψ_c. In other words $\Psi_c := \psi_c(\phi_c(1_c))$. Now that we have a cocone Ψ by the universal property of the colimit there exists a unique natural transformation $\operatorname{colim} F_\gamma \Rightarrow \nabla_Y$ so that

$$\begin{tikzcd} F_\gamma \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\Psi"'] & \operatorname{colim} F_\gamma \arrow[dl, Rightarrow, dashed] \\ \nabla_Y & \end{tikzcd}$$

commutes. In terms of X we have a unique map $\mathrm{Re}(X) \longrightarrow Y$. We leave it to the reader to finish the pf. ∎

<u>Remark</u> Recall that if a right adjoint exists it is unique up to unique natural isomorphism.

<u>Example</u> Let $\mathcal{C} = \Delta$ (see my notes on simplicial sets) and let $\Delta \xrightarrow{\gamma} \mathrm{Top}$ send $\underline{n} \longmapsto \Delta^n$, the standard topological n-simplex, and the face & degeneracy maps get sent to the obvious maps (think about it). Then $\mathrm{Re}(X) = |X|$. In fact, this was the definition given in our earlier lectures! As for Sing, let Y be a topological space. Then $\mathrm{Sing}(Y)$ sends $\underline{n}$ to $\mathrm{Top}(\Delta^n, Y)$ which exactly gives the usual singular simplicial set (the morphisms get sent to the usual face and degeneracy maps). Therefore, as a consequence of the previous Theorem, we know that $|\cdot| \dashv \mathrm{Sing}$.

<u>Def'n</u> Recall, a <u>model category</u> is a category equipped with three classes of morphisms called fibrations, cofibrations, and weak equivalences satisfying several axioms which we will not reproduce here.

<u>Example</u> In our previous lecture we gave the category of simplicial sets a model structure by declaring the fibrations to be

Kan fibrations, the cofibrations to be inclusions, and the weak equivalences to be maps that are weak equivalences in CGHaus ($\subset$ Top) after applying realization.

<u>Example</u> The category of simplicial presheaves over a category $\mathcal{C}$ (i.e. $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{sSet}) \equiv \mathrm{sSet}^{\mathcal{C}^{op}}$) has a model structure by setting a map
$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathrm{sSet} \arrow[from=U, to=D, Rightarrow, "\psi"] \end{tikzcd}$$
to be a

i) fibration if $Fc \xrightarrow{\psi_c} Gc$ is a Kan fibration for every c in $\mathcal{C}^{op}$

ii) weak equivalence if $Fc \xrightarrow{\psi_c} Gc$ is a weak equivalence in sSet for every c in $\mathcal{C}^{op}$

iii) cofibration if it has the left-lifting property w.r.t. fibrations that are also weak equivalences.

<u>Defn</u> Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A <u>Quillen pair</u> is an adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$ with L left adjoint to R such that L preserves weak equivalences between cofibrant objects (or equivalently R preserves weak equivalences between fibrant objects).

<u>Defn</u> Let $\mathcal{C}$ be a category, $\mathcal{M}$ and $\mathcal{N}$ model categories, and functors as in
$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & \mathcal{M} \\ & \mathcal{N} \end{tikzcd}$$
A <u>factorization of γ through $\mathcal{M}$</u> consists of

(i) a Quillen pair $\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "L"] & \mathcal{N} \arrow[l, shift left, "R"] \end{tikzcd}$ and

(ii) a natural weak equivalence η as in
$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & \mathcal{M} \arrow[d, "L"] \arrow[dl, phantom, "\Swarrow\eta"] \\ & \mathcal{N} \end{tikzcd}.$$

A factorization as above will be denoted by (L, R, η). A morphism $(L,R,\eta) \longrightarrow (L',R',\eta')$ is a natural transformation $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "L"{name=U}] \arrow[r, bend right, "L'"'{name=D}] & \mathcal{N} \arrow[from=U, to=D, Rightarrow, "\psi"] \end{tikzcd}$ so that
$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, bend right, "\gamma"'] \arrow[dr, phantom, "\Swarrow\eta'"] & \mathcal{M} \arrow[d, bend right, "L'"'{name=A}] \arrow[d, bend left, "L"{name=B}] \arrow[from=B, to=A, phantom, "\Leftarrow\psi"] \\ & \mathcal{N} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & \mathcal{M} \arrow[d, "L"] \arrow[dl, phantom, "\Swarrow\eta"] \\ & \mathcal{N} \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} Lr \arrow[rr, Rightarrow, "\psi r"] \arrow[dr, Rightarrow, "\eta"'] & \arrow[d, phantom, "="] & L'r \arrow[dl, Rightarrow, "\eta'"] \\ & \gamma & \end{tikzcd}.$$

<u>Defn</u> A category $\mathcal{C}$ is <u>contractible</u> if $|\mathcal{N}\mathcal{C}|$ is contractible in Top (here $\mathcal{N}$ is the nerve functor $\mathcal{N}: \mathrm{Cat} \to \mathrm{sSet}$ and $|\cdot|$ is the geometric realization written as $\mathrm{Re}: \mathrm{sSet} \longrightarrow \mathrm{Top}$ from our earlier example).

<u>Theorem</u> Let $\mathcal{C}$ be a small category. Then there exists a model category $\mathcal{U}\mathcal{C}$ together with a functor $\mathcal{C} \xrightarrow{r} \mathcal{U}\mathcal{C}$ satisfying the following universal property: For any functor $\mathcal{C} \xrightarrow{\gamma} \mathcal{M}$ into a model category $\mathcal{M}$ there exists a factorization of γ through $\mathcal{U}\mathcal{C}$. Moreover, the category of such factorizations is contractible.

<u>Remark</u> Contractibility of the category of factorizations is the analogue/generalization of uniqueness to the setting of homotopy.

<u>proof sketch</u> Let $\mathcal{U}\mathcal{C} := \mathrm{sSet}^{\mathcal{C}^{op}}$, simplicial presheaves over $\mathcal{C}$. The proof is similar to the analogue in ordinary category theory but involves homotopy colimits, a topic of future discussion.