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Theorem Let \mathcal{C} be a small category. Then there exists a co-complete category (a category that has all small colimits) \mathcal{D} together with a functor $\mathcal{C} \xrightarrow{\gamma} \mathcal{D}$ satisfying the following universal property:
 For any co-complete category \mathcal{D}' with a functor $\mathcal{C} \xrightarrow{\gamma'} \mathcal{D}'$ there exists a colimit-preserving functor $\mathcal{D} \xrightarrow{Re} \mathcal{D}'$ so that $\mathcal{C} \xrightarrow{\gamma} \mathcal{D} \xrightarrow{Re} \mathcal{D}'$ commutes. Moreover, any two such functors $\mathcal{D} \rightarrow \mathcal{D}'$ are canonically isomorphic.

Furthermore, Re has a right adjoint $\mathcal{D}' \xrightarrow{Sing} \mathcal{D}$.

proof/construction Let $\mathcal{E} := \text{Pre}(\mathcal{C})$ the category of presheaves on \mathcal{C} with values in Set and let $\mathcal{C} \xrightarrow{r} \text{Pre}(\mathcal{C})$ be the Yoneda embedding defined by

$$X \mapsto \left(\begin{array}{l} r(X) : Y \mapsto \mathcal{C}(Y, X) \text{ on objects} \\ \& (Y \xrightarrow{f} Z) \mapsto (\mathcal{C}(Z, X) \xrightarrow{of} \mathcal{C}(Y, X)) \text{ on morphisms} \end{array} \right)$$

$$\text{and } (X \xrightarrow{\psi} X') \mapsto \left(\begin{array}{l} r(\psi) : r(X) \Rightarrow r(X') \\ Y \mapsto (\mathcal{C}(Y, X) \xrightarrow{\psi_{\circ}} \mathcal{C}(Y, X')) \end{array} \right)$$

on morphisms. The Yoneda Lemma states that this Yoneda embedding is full and faithful i.e. $\mathcal{C}(X, X') \cong \text{Pre}(\mathcal{C})(r(X), r(X'))$ canonically. Now let's prove that $\mathcal{C} \xrightarrow{r} \text{Pre}(\mathcal{C})$ satisfies the required universal property. Let $\mathcal{C} \xrightarrow{\gamma} \mathcal{D}$

be as above. To construct Re we will only use γ and the fact that objects of $\text{Pre}(\mathcal{C})$ are colimits of representables. More precisely, we recall the Density Theorem as a Lemma.

Lemma Let X be a presheaf of sets on \mathcal{C} . Let $\mathcal{C} \downarrow X$ be the category whose objects are pairs $(c, r(c) \rightarrow X)$ in $\mathcal{C} \& \text{Pre}(\mathcal{C})$ respectively and whose morphisms are morphisms $r(c) \rightarrow r(c')$ (or equivalently, by the Yoneda Lemma, $r(c) \rightarrow r(c')$) making $r(c) \rightarrow r(c') \rightarrow X$ commute. Define a diagram in $\text{Pre}(\mathcal{C})$ by the functor

$$\mathcal{C} \downarrow X \xrightarrow{F} \text{Pre}(\mathcal{C}) \text{ that sends } \left(c, \begin{array}{c} r(c) \\ \downarrow \\ X \end{array} \right) \mapsto r(c) \text{ on objects and } (c \rightarrow c') \mapsto (r(c) \rightarrow r(c')) \text{ on morphisms.}$$

Then $X \cong \text{colim} F \cong \text{colim}_{c \in \mathcal{C}, \mathcal{C} \downarrow X} r(c)$. We won't prove this. Using this, we define Re to be $Re(X) := \text{colim}_{c \in \mathcal{C}, \mathcal{C} \downarrow X} \gamma(c) = \text{colim} F_{\gamma}$, where

$$\mathcal{C} \downarrow X \xrightarrow{F_{\gamma}} \mathcal{D} \text{ is defined by } \left(c, \begin{array}{c} r(c) \\ \downarrow \\ X \end{array} \right) \mapsto \gamma(c) \text{ on objects and } (c \rightarrow c') \mapsto (\gamma(c) \rightarrow \gamma(c')) \text{ on morphisms.}$$

This is technically not a functor but this issue can be taken care of. Because colimits are unique up to canonical isomorphism, any two such functors $\text{Pre}(\mathcal{C}) \xrightarrow{Re} \mathcal{D}$ that agree on \mathcal{C} via γ , agree up to such a natural isomorphism. Now, define $\mathcal{D} \xrightarrow{Sing} \text{Pre}(\mathcal{C})$ to be the assignment

$d \mapsto \left(\text{Sing}(d) : C \mapsto \mathcal{D}(\gamma_C, d) \text{ on objects} \right)$
 $\left(c \xrightarrow{f} c' \right) \mapsto \left(\mathcal{D}(\gamma_{c'}, d) \xrightarrow{-\circ f} \mathcal{D}(\gamma_c, d) \right)$
 on morphisms

on objects and

$(d \xrightarrow{\psi} d') \mapsto \left(\text{Sing}(\psi) : \text{Sing}(d) \Rightarrow \text{Sing}(d') \right)$
 $c \mapsto \left(\mathcal{D}(\gamma_c, d) \xrightarrow{\psi \circ -} \mathcal{D}(\gamma_c, d') \right)$

on morphisms. To check that Sing

is a right adjoint of Re we need to show $\mathcal{D}(\text{Re}(X), Y) \cong \underline{\mathcal{C}}(X, \text{Sing}(Y))$ is an isomorphism natural in X and Y . We'll

construct the map from right to left. Let $X \xrightarrow{\psi} \text{Sing}(Y)$ be a natural transformation.

So to every c we get a function $X(c) \xrightarrow{\psi_c} \mathcal{D}(\gamma_c, Y)$ satisfying naturality for all $c \xrightarrow{f} c'$. Now, a morphism

$\text{Re}(X) = \text{colim}_{c \in C, e \in X} \delta(c) \longrightarrow Y$ is specified by a collection of compatible

morphisms $\delta(c) \longrightarrow Y$ by the universal property of the colimit. The natural transformation $X \xrightarrow{\psi} \text{Sing}(Y)$ defines such a

cocone $\mathcal{C} \downarrow X \begin{array}{c} \xrightarrow{F_X} \mathcal{D} \\ \Downarrow \Psi \\ \Delta_Y \end{array}$ sending $\left(c, \begin{array}{c} r(c) \\ \downarrow \phi \\ X \end{array} \right)$

to a morphism $\delta(c) \xrightarrow{\psi_c} Y$. To see what

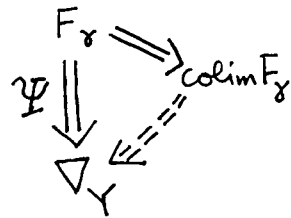
this morphism should be look at the image of 1_c under $r(c)(c) = \mathcal{C}(c, c) \xrightarrow{\phi_c} X(c)$ and post compose with ψ_c . In other

words $\Psi_c := \psi_c(\phi_c(1_c))$. Now that we have a cocone Ψ by the universal property of the colimit there exists

a unique natural transformation $\text{colim } F_Y \Rightarrow \Delta_Y$ so that

commutes. In terms of X we have a unique map $\text{Re}(X) \longrightarrow Y$. We

leave it to the reader to finish the pf. \blacksquare



Remark Recall that if a right adjoint exists it is unique up to unique natural isomorphism.

Example Let $\mathcal{C} = \Delta$ (see my notes on simplicial sets) and let $\Delta \xrightarrow{\gamma} \text{Top}$ send $\underline{n} \mapsto \Delta^n$, the standard topological n -simplex, and the face & degeneracy maps get sent to the obvious maps (think about it). Then $\text{Re}(X) = |X|$. In fact, this was the definition given in our earlier lectures! As for Sing , let Y be a topological space. Then $\text{Sing}(Y)$ sends \underline{n} to $\text{Top}(\Delta^n, Y)$ which exactly gives the usual singular simplicial set (the morphisms get sent to the usual face and degeneracy maps). Therefore, as a consequence of the previous theorem, we know that $|-| \dashv \text{Sing}$.

Def'n Recall, a model category is a category equipped with three classes of morphisms called fibrations, cofibrations, and weak equivalences satisfying several axioms which we will not reproduce here.

Example In our previous lecture we gave the category of simplicial sets a model structure by declaring the fibrations to be

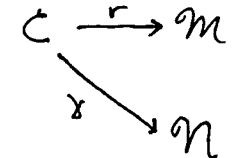
Kan fibrations, the cofibrations to be inclusions, and the weak equivalences to be maps that are weak equivalences in $CGHws$ ($CTop$) after applying realization.

Example The category of simplicial presheaves over a category \mathcal{C} (i.e. $Fun(\mathcal{C}^{op}, sSet) \cong sSet^{\mathcal{C}^{op}}$) has a model structure by setting a map $\mathcal{C}^{op} \xrightarrow{F} sSet$ to be a

- i) fibration if $Fc \xrightarrow{\gamma_c} Gc$ is a Kan fibration for every c in \mathcal{C}^{op}
- ii) weak equivalence if $Fc \xrightarrow{\gamma_c} Gc$ is a weak equivalence in $sSet$ for every c in \mathcal{C}^{op}
- iii) cofibration if it has the left-lifting property w.r.t. fibrations that are also weak equivalences.

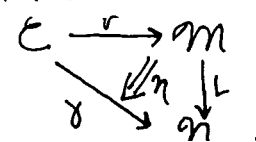
Defin Let \mathcal{C} and \mathcal{D} be model categories. A Quillen pair is an adjunction $\mathcal{C} \xrightleftharpoons[L]{L} \mathcal{D}$ with L left adjoint to R such that L preserves weak equivalences between cofibrant objects (or equivalently R preserves weak equivalences between fibrant objects).

Defin Let \mathcal{C} be a category, \mathcal{M} and \mathcal{N} model categories, and functors as in

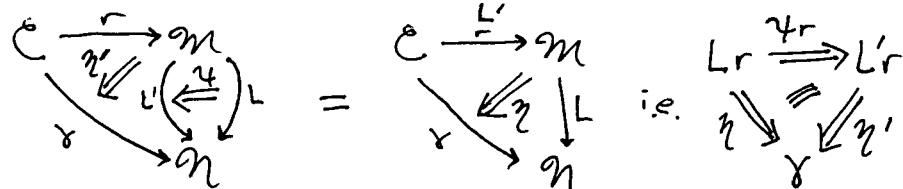


A factorization of γ through \mathcal{M} consists of

- (i) a Quillen pair $\mathcal{M} \xrightleftharpoons[L]{L} \mathcal{N}$ and
- (ii) a natural weak equivalence η as in



A factorization as above will be denoted by (L, R, η) . A morphism $(L, R, \eta) \rightarrow (L', R', \eta')$ is a natural transformation $\mathcal{M} \xrightarrow{L} \mathcal{N} \xrightarrow{L'} \mathcal{N}$ so that



Defin A category \mathcal{C} is contractible if $|N\mathcal{C}|$ is contractible in Top (here N is the nerve functor $N: Cat \rightarrow sSet$ and $|\cdot|$ is the geometric realization written as $Re: sSet \rightarrow Top$ from our earlier example).

Theorem Let \mathcal{C} be a small category. Then there exists a model category $\mathcal{U}\mathcal{C}$ together with a functor $\mathcal{C} \xrightarrow{r} \mathcal{U}\mathcal{C}$ satisfying the following universal property: For any functor $\mathcal{C} \xrightarrow{\gamma} \mathcal{M}$ into a model category \mathcal{M} there exists a factorization of γ through $\mathcal{U}\mathcal{C}$. Moreover, the category of such factorizations is contractible.

Remark Contractibility of the category of factorizations is the analogue/generalization of uniqueness to the setting of homotopy. proof sketch let $\mathcal{U}\mathcal{C} := sSet^{\mathcal{C}^{op}}$, simplicial presheaves over \mathcal{C} . The proof is similar to the analogue in ordinary category theory but involves homotopy colimits, a topic of future discussion.