Notes on the Gelfand-Naimark theorem

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Abstract

These are some notes on the Gelfand-Naimark theorem. Briefly, one version of the Gelfand-Naimark theorem states that the category of locally compact Hausdorff spaces is equivalent to the category of commutative Banach algebras. We will describe this equivalence, beginning with one for compact Hausdorff spaces, in category-theoretic language. Absolutely nothing here is novel and is just a rewriting of Chapter 1 of Folland's exposition in [1]. I will only prove things that cannot be found in this book. The only difference is the language used. This current version is an expansion of an earlier one from June 2015. Any errors/misinterpretations are solely due to me.

1 C^*-Algebras and the spectrum

Definition 1.1. A *C^*-algebra* is a Banach algebra $\mathcal{A}$ over the complex numbers $\mathbb{C}$ equipped with an *involution*, a linear anti-automorphism

$$\begin{aligned} \mathcal{A} &\to \mathcal{A} \\ x &\mapsto x^* \end{aligned} \tag{1.2}$$

that squares to the identity, such that the involution satisfies

$$\|x^*x\| = \|x\|^2 \tag{1.3}$$

for all $x \in \mathcal{A}$. If a C^*-algebra $\mathcal{A}$ has a unit e that satisfies e^*, then $\mathcal{A}$ is a *unital C^*-algebra*. If no such unit exists, then $\mathcal{A}$ is a *non-unital C^*-algebra*. If a C^*-algebra $\mathcal{A}$ satisfies $xy = yx$ for all $x, y \in \mathcal{A}$, then $\mathcal{A}$ is a *commutative C^*-algebra*. The words "unital," "non-unital," and "commutative" are adjectives so that, for instance, the term *unital commutative C^*-algebra* is a combination of the two previous ones.

The following four examples suffice for our purposes. The first one is the most basic example. The two that follow suffice for the statements of the Gelfand-Naimark theorem and the fourth is relevant for an extension of this theorem to the "non-commutative" setting of the Gelfand-Naimark-Segal theorem.

Example 1.4. $\mathbb{C}$ with its usual structure is a unital commutative C^*-algebra.

Example 1.5. Let X be a compact Hausdorff space. Then the space of complex-valued continuous functions $C(X)$ with

- pointwise addition as the addition,
- pointwise multiplication as the product,
- norm given by
$$\|f\| := \sup_{x \in X} |f(x)|, \qquad f \in C(X), \tag{1.6}$$
- complex conjugation as the involution,
- and the constant function whose value is 1 at every point

is a unital commutative C^*-algebra.

Example 1.7. Let X be a noncompact locally compact Hausdorff space. Then the subspace of compactly-supported complex-valued continuous functions $C_c(X) \subset C(X)$ is a non-unital commutative C^*-algebra.

Example 1.8. Let $\mathcal{H}$ be a Hilbert space. Then the Banach space of bounded linear operators $\mathcal{B}(\mathcal{H})$ with the operator norm and involution given by sending an operator to its adjoint is a unital commutative C^*-algebra.

Definition 1.9. Let $\mathcal{A}$ be a commutative Banach algebra. A *character* on $\mathcal{A}$ is a nonzero morphism $\chi : \mathcal{A} \to \mathbb{C}$ of Banach algebras.

Characters form a subset of $\mathcal{A}^\vee$, the dual space of $\mathcal{A}$. This dual space has a natural topology coming from the norm. However, the weak* topology on $\mathcal{A}^\vee$ is also relevant for some purposes so we briefly review it.

Construction 1.10. For each $x \in \mathcal{A}$, let $\| \cdot \|_x : \mathcal{A}^\vee \to \mathbb{C}$ be the semi-norm[1] defined by
$$\mathcal{A}^\vee \ni \chi \mapsto \|\chi\|_x := |\chi(x)|, \qquad \text{for } \chi \in \mathcal{A}^\vee. \tag{1.11}$$
For each $\chi \in \mathcal{A}^\vee, \epsilon > 0, n \in \mathbb{N}$, and $\alpha_n : \{1, \dots, n\} \equiv \overline{n} \to \mathcal{A}$, let
$$B_\epsilon^{\alpha_n}(\chi) := \{\xi \in \mathcal{A}^\vee \mid \|\chi - \xi\|_{\alpha_n(j)} < \epsilon, j \in \overline{n}\}. \tag{1.12}$$
Then
$$\{B_\epsilon^{\alpha_n}(\chi) \mid \epsilon > 0, n \in \mathbb{N}, \alpha_n : \overline{n} \to \mathcal{A}, \chi \in \mathcal{A}^\vee\} \tag{1.13}$$
is a basis[2] for a topology on $\mathcal{A}^\vee$. This topology is known as the *weak* topology* on $\mathcal{A}^\vee$.

[1] A *semi-norm* on a vector space V is a function $\| \cdot \| : V \to \mathbb{C}$ satisfying

i) $\|v\| \geq 0$ for all $v \in V$,

ii) $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{C}, v \in V$, and

iii) $\|v + w\| \leq \|v\| + \|w\|$.

[2] Recall, a *basis* on a set X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X satisfying

Definition 1.14. Let $\mathcal{A}$ be a commutative Banach algebra. The *spectrum of $\mathcal{A}$* is the set

$$\sigma(\mathcal{A}) := \{\chi : \mathcal{A} \to \mathbb{C} \mid \chi \text{ is a character}\}. \tag{1.15}$$

equipped with the subspace topology coming from $\mathcal{A}^\vee$ via the weak* topology.

Proposition 1.16. *Let $\mathcal{A}$ be a commutative Banach algebra. Then the spectrum $\sigma(\mathcal{A})$ is a nonempty locally compact Hausdorff space. Furthermore, $\sigma(\mathcal{A})$ is compact if and only if $\mathcal{A}$ is unital.*

Proof. See Theorem 1.30 in [1] for the case that $\mathcal{A}$ is not necessarily unital. The second statement follows from the following (see comments after Proposition 1.10 in [1]). First, recall Alaoglu's theorem, which states that the closed unit ball in $\mathcal{A}^\vee$ is compact in the weak* topology. If $\mathcal{A}$ is unital, then $\sigma(\mathcal{A})$ is a closed subset of this unit ball and is therefore compact. Conversely, suppose $\sigma(\mathcal{A})$ is compact. For each $x \in \mathcal{A}$, let U_x be the subset of $\sigma(\mathcal{A})$ given by

$$U_x := \{\chi \in \sigma(\mathcal{A}) \mid \chi(x) \neq 0\}. \tag{1.17}$$

Notice that it satisfies the condition

$$U_{xy} = U_x \cap U_y \text{ for all } x, y \in \mathcal{A}. \tag{1.18}$$

U_x is also an open subset[3] for each $x \in \mathcal{A}$. Furthermore, the collection $\{U_x\}_{x\in\mathcal{A}}$ is an open cover of $\mathcal{A}$. By compactness, there exists an open subcover $\mathcal{U} := \{U_{x_1}, \ldots, U_{x_n}\}$. Let

$$z := \sum_{i=1}^{n} x_i x_i^*. \tag{1.19}$$

Then for *any* $\chi \in \sigma(\mathcal{A})$, $\chi(z) > 0$ because

$$\chi(z) = \chi\left(\sum_{i=1}^{n} x_i x_i^*\right) = \sum_{i=1}^{n} |\chi(x_i)|^2 > 0 \tag{1.20}$$

because *at least* one of the x_i's satisfies $\chi(x_i) \neq 0$ since $\mathcal{U}$ is a cover of $\sigma(\mathcal{A})$. In particular, $U_z = \sigma(\mathcal{A})$. If we show that z is an invertible element in $\mathcal{A}$, then we will be done. ∎

Finish this proof. Why is $\mathcal{A}$ unital? Can I construct the unit explicitly, or do I have to use some non-constructive argument?

Proposition 1.21. *Let $\mathcal{A} \xrightarrow{f} \mathcal{A}'$ be a morphism of commutative Banach algebras. Then the function*

$$\begin{aligned} \sigma(\mathcal{A}') &\xrightarrow{\sigma(f)} \sigma(\mathcal{A}) \\ \chi' &\mapsto \chi' \circ f \end{aligned} \tag{1.22}$$

is continuous.

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i) for each $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$,

ii) for every pair $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists a $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

A basis $\mathcal{B}$ as above can be used to construct a topology on X by setting τ to be the smallest topology containing $\mathcal{B}$. Explicitly,

$$\tau_{\mathcal{B}} = \left\{ \bigcup_\alpha B_\alpha \mid B_\alpha \in \mathcal{B} \right\}.$$

This topology is called the *topology generated* by $\mathcal{B}$.

[3] This is not obvious. One should check this. At some point, I will include a proof here.

Proof. First, note that since

$$(\chi' \circ f)(xy) = \chi'\big(f(x)f(y)\big) = \chi'\big(f(x)\big)\chi'\big(f(y)\big) = (\chi' \circ f)(x)(\chi' \circ f)(y) \tag{1.23}$$

for all $x, y \in \mathcal{A}$, $\chi' \circ f$ is indeed an element of $\sigma(\mathcal{A})$. To see that $\sigma(f)$ is continuous, it suffices to show that the inverse image of a basis element gets sent to an open set. By definition of the subspace topology, a basis for the topology on $\sigma(\mathcal{A})$ is given by

$$\sigma(\mathcal{A}) \cap B_{\epsilon}^{\alpha_n}(\xi) = \big\{\chi \in \sigma(\mathcal{A}) \mid \|\chi - \xi\|_{\alpha_n(j)} < \epsilon, j \in \overline{n}\big\} \tag{1.24}$$

over all $\xi \in \mathcal{A}^{\vee}, \alpha_n : \overline{n} \to \mathcal{A}, \epsilon > 0, n \in \mathbb{N}$. $\sigma(f)$ is continuous if[4] for every $\chi' \in \sigma(\mathcal{A}')$ and every basic set[5] $\sigma(\mathcal{A}) \cap B_{\epsilon}^{\alpha_n}(\chi' \circ f)$ there exists an open set containing χ' whose image is contained in this set. In fact, the basic set $B_{\epsilon}'^{f\circ\alpha_n}(\chi')$ accomplishes this goal because

$$\|\chi' - \xi'\|_{f\circ\alpha_n(j)} = \Big|\chi'\Big(f\big(\alpha_n(j)\big)\Big) - \xi'\Big(f\big(\alpha_n(j)\big)\Big)\Big| = \|\chi' \circ f - \xi' \circ f\|_{\alpha_n(j)}, \tag{1.25}$$

which shows that

$$\begin{aligned} \sigma(f)\Big(\sigma(\mathcal{A}') \cap B_{\epsilon}'^{f\circ\alpha_n}(\chi')\Big) &= \Big\{\xi' \circ f \in \sigma(\mathcal{A}) \mid \|\chi' - \xi'\|_{f\circ\alpha_n(j)} < \epsilon, j \in \overline{n}\Big\} \\ &= \Big\{\xi' \circ f \in \sigma(\mathcal{A}) \mid \|\chi' \circ f - \xi' \circ f\|_{\alpha_n(j)} < \epsilon, j \in \overline{n}\Big\} \\ &\subseteq \sigma(\mathcal{A}) \cap B_{\epsilon}^{\alpha_n}(\chi' \circ f). \end{aligned} \tag{1.26}$$

Thus, $\sigma(f)$ is continuous. ■

2 The Gelfand-Naimark equivalence

Let's use the following terminology to describe the categories of interest

Category	Notation
commutative C^*-algebras	**cC*-Alg**
unital commutative C^*-algebras	**ucC*-Alg**
locally compact Hausdorff spaces	**lcHaus**
compact Hausdorff spaces	**cHaus**

Proposition 2.1. *The assignment*

$$\begin{aligned} \mathbf{cC^*\text{-}Alg}^{\mathrm{op}} &\xrightarrow{\sigma} \mathbf{lcHaus} \\ \mathcal{A} &\mapsto \sigma(\mathcal{A}) \\ \big(\mathcal{A} \xrightarrow{f} \mathcal{A}'\big) &\mapsto \Big(\sigma(\mathcal{A}) \xleftarrow{\sigma(f)} \sigma(\mathcal{A}')\Big) \end{aligned} \tag{2.2}$$

from Definition 1.14 and Proposition 1.21 defines a functor. A similar statement holds for $\mathbf{ucC^*\text{-}Alg}^{\mathrm{op}} \xrightarrow{\sigma} \mathbf{cHaus}$.

[4] Let $f : X \to X'$ be a function and let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies on X and X', respectively. f is continuous if for every $x \in X$, and for any $B' \in \mathcal{B}'$ with $f(x) \in B'$, there exists a $B \in \mathcal{B}$ with $x \in B$ such that $f(B) \subset B'$.

[5] Technically, we should have said that for every $\chi' \in \sigma(\mathcal{A}')$ and every basic set B containing $\chi' \circ f$, there exists an open set U' containing χ' whose image is contained in B. However, due to the semi-norms, it suffices to take the basic sets B containing $\chi' \circ f$ to be "centered" at $\chi' \circ f$.

Proof. Exercise. ■

Proposition 2.3. *The assignment*

$$
\begin{aligned}
\mathbf{lcHaus} &\xrightarrow{C_c} \mathbf{cC^*\text{-}Alg}^{\mathrm{op}} \\
X &\mapsto C_c(X) \\
\left(X \xrightarrow{f} Y\right) &\mapsto \left(X \xleftarrow{C_c(f)} Y\right)
\end{aligned} \tag{2.4}
$$

where $C_c(f)$ takes a compactly supported continuous function $\varphi : Y \to \mathbb{C}$ to $\varphi \circ f$, defines a functor. A similar statement holds for $\mathbf{cHaus} \xrightarrow{C} \mathbf{ucC^*\text{-}Alg}^{\mathrm{op}}$.

Proof. Exercise. ■

Therefore, we have defined functors

$$
\mathbf{ucC^*\text{-}Alg}^{\mathrm{op}} \underset{C}{\overset{\sigma}{\rightleftarrows}} \mathbf{cHaus} \qquad \& \qquad \mathbf{cC^*\text{-}Alg}^{\mathrm{op}} \underset{C_c}{\overset{\sigma}{\rightleftarrows}} \mathbf{lcHaus}\,. \tag{2.5}
$$

We will show that these functors are part of an adjoint equivalence of categories by constructing natural isomorphisms

$$
\Gamma : \mathrm{id} \Rightarrow C_c \circ \sigma \qquad \& \qquad \mathrm{ev} : \mathrm{id} \Rightarrow \sigma \circ C_c \tag{2.6}
$$

and similarly for the compact/unital case. The natural transformation Γ is known as the *Gelfand transform*. The natural transformation ev is just an evaluation map as will be shown shortly.

Definition 2.7. Let $\mathcal{A}$ be a commutative Banach algebra and define

$$
\begin{aligned}
\Gamma_{\mathcal{A}} : \mathcal{A} &\to C_c(\sigma(\mathcal{A})) \\
x &\mapsto \Big(\chi \mapsto (\Gamma_{\mathcal{A}}(x))(\chi) := \chi(x)\Big).
\end{aligned} \tag{2.8}
$$

$\Gamma_{\mathcal{A}}$ is called the *Gelfand transform on $\mathcal{A}$*.

Proposition 2.9. *The following facts are true regarding the Gelfand transform.*

i) For every commutative Banach algebra $\mathcal{A}$, the Gelfand transform $\Gamma_{\mathcal{A}} : \mathcal{A} \to C_c(\sigma(\mathcal{A}))$ is a morphism of Banach algebras.

ii) If $\mathcal{A}$ is in additional unital, then the Gelfand transform $\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\sigma(\mathcal{A}))$ is a continuous unital algebra map.

iii) If $\mathcal{A}$ is a commutative C^*-algebra, then $\Gamma_{\mathcal{A}}(x^*) = \big(\Gamma_{\mathcal{A}}(x)\big)^*$. Hence, $\Gamma_{\mathcal{A}}$ is a morphism of C^*-algebras.

iv) If $\mathcal{A}$ is a commutative unital C^*-algebra, then $\Gamma_{\mathcal{A}}$ is a unital C^*-algebra isomorphism (in particular, it is isometric).

v) If $\mathcal{A}$ is a commutative C^*-algebra, then $\Gamma_{\mathcal{A}}$ is a C^*-algebra isomorphism.

Proof.

i) See Theorem 1.13 parts (a) and (d) in [1].

ii) See Theorem 1.13 part (a) in [1].

iii) See Proposition 1.14 in [1].

iv) See Theorem 1.20 in [1].

v) See Theorems 1.30 and 1.31 in [1].

■

Remark 2.10. If $\mathcal{A}$ is just an involutive Banach algebra, then iii) fails.

Proposition 2.11. *The assignment*

$$\begin{gathered} \mathbf{cC^*\text{-}Alg}_0^{\mathrm{op}} \xrightarrow{\Gamma} \mathbf{cC^*\text{-}Alg}_1^{\mathrm{op}} \\ \mathcal{A} \mapsto \Big(\mathcal{A} \xrightarrow{\Gamma_{\mathcal{A}}} C_c\big(\sigma(\mathcal{A})\big)\Big) \end{gathered} \tag{2.12}$$

from Definition 2.7 defines a natural isomorphism $\Gamma : \mathrm{id}_{\mathbf{cC^*\text{-}Alg}^{\mathrm{op}}} \Rightarrow C_c \circ \sigma$.

Proof. For any morphism $\mathcal{A} \xrightarrow{f} \mathcal{A}'$ of C^*-algebras, we have the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma_{\mathcal{A}}} & C_c\big(\sigma(\mathcal{A})\big) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle C_c(\sigma(f))} \\ \mathcal{A}' & \xrightarrow[\Gamma_{\mathcal{A}'}]{} & C_c\big(\sigma(\mathcal{A}')\big) \end{array} . \tag{2.13}$$

To see that it commutes, let $x \in \mathcal{A}$. Going along the top and right arrows gives

$$\Big(C_c\big(\sigma(f)\big)\Big)\Big(\Gamma_{\mathcal{A}}(x)\Big) = \Gamma_{\mathcal{A}}(x) \circ \sigma(f) \tag{2.14}$$

by definition of C_c as a functor (see Proposition 2.3). Going along the left and bottom arrows gives

$$\Gamma_{\mathcal{A}'}\big(f(x)\big). \tag{2.15}$$

To see that these two elements of $C_c\big(\sigma(\mathcal{A}')\big)$ are equal, let $\chi' \in \sigma(\mathcal{A}')$. Then

$$\begin{array}{ccc} \Gamma_{\mathcal{A}}(x)\Big(\sigma(f)\big(\chi'\big)\Big) & & \Gamma_{\mathcal{A}'}\big(f(x)\big)\big(\chi'\big) \\ \| {\scriptstyle (1.22)} & & \| {\scriptstyle (2.8)} \\ \Gamma_{\mathcal{A}}(x)\big(\chi' \circ f\big) & & \chi'\big(f(x)\big) \\ {\scriptstyle (2.8)} \| & & \| \\ & (\chi' \circ f)(x) & \end{array} , \tag{2.16}$$

which proves that Γ is a natural transformation. It is a natural isomorphism by v) of Proposition 2.9. ■

Definition 2.17. Let X be a locally compact Hausdorff space. The *evaluation map* is the assignment

$$\begin{aligned} \mathrm{ev}_X : X &\to \sigma\big(C_c(X)\big) \\ x &\mapsto \Big(f \mapsto \big(\mathrm{ev}_X(x)\big)(f) := f(x)\Big). \end{aligned} \tag{2.18}$$

Proposition 2.19. *For every locally compact Hausdorff space X, ev_X is a homeomorphism.*

Proof. See Theorem 1.16 of [1] for the compact case. F X non-compact, see Example 2 after Theorem 1.30 in [1]. ∎

Proposition 2.20. *The assignment*

$$\begin{aligned} \mathbf{lcHaus}_0 &\xrightarrow{\mathrm{ev}} \mathbf{lcHaus}_1 \\ X &\mapsto \Big(X \xrightarrow{\mathrm{ev}_X} \sigma\big(C_c(X)\big)\Big) \end{aligned} \tag{2.21}$$

from Definition 2.17 defines a natural isomorphism $\mathrm{ev} : \mathrm{id}_{\mathbf{lcHaus}} \Rightarrow \sigma \circ C_c$.

Proof. For any morphism $X \xrightarrow{f} Y$ of locally compact Hausdorff spaces, we have the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{ev}_X} & \sigma\big(C_c(X)\big) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma(C_c(f))} \\ Y & \xrightarrow[\mathrm{ev}_Y]{} & \sigma\big(C_c(Y)\big) \end{array} . \tag{2.22}$$

To see that this diagram commutes, let $x \in X$. Going along the top and right arrows gives

$$\Big(\sigma\big(C_c(f)\big)\Big)\Big(\mathrm{ev}_X(x)\Big) = \mathrm{ev}_X(x) \circ C_c(f) \tag{2.23}$$

by definition of σ as a functor (see Proposition 1.21). Going along the left and bottom arrows gives

$$\mathrm{ev}_Y\big(f(x)\big). \tag{2.24}$$

To see that these two elements of $\sigma\big(C_c(Y)\big)$, i.e. non-zero characters $C_c(Y) \to \mathbb{C}$, are equal, let $\varphi : Y \to \mathbb{C}$ be a compactly supported continuous function on Y. Then

$$\begin{array}{ccc} \mathrm{ev}_X(x)\Big(C_c(f)(\varphi)\Big) & & \mathrm{ev}_Y\big(f(x)\big)\Big(\varphi\Big) \\ \| {\scriptstyle (2.4)} & & \| {\scriptstyle (2.18)} \\ \mathrm{ev}_X(x)\big(\varphi \circ f\big) & & \varphi\big(f(x)\big) \\ & {\scriptstyle (2.18)}\searrow\!\!= \quad =\!\!\swarrow & \\ & (\varphi \circ f)(x) & \end{array} , \tag{2.25}$$

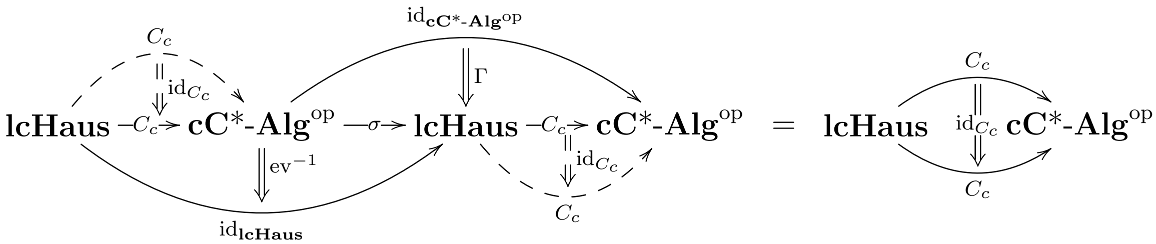
which proves that ev is a natural transformation. It is a natural isomorphism by Proposition 2.19. ∎

Theorem 2.26. *Using the notation from above*

$$\left(\mathbf{cC^*\text{-}Alg}^{\mathrm{op}} \xrightarrow{\sigma} \mathbf{lcHaus}, \mathbf{lcHaus} \xrightarrow{C_c} \mathbf{cC^*\text{-}Alg}^{\mathrm{op}}, \mathrm{id} \overset{\Gamma}{\Rightarrow} C_c \circ \sigma, \sigma \circ C_c \xRightarrow{\mathrm{ev}^{-1}} \mathrm{id}\right) \tag{2.27}$$

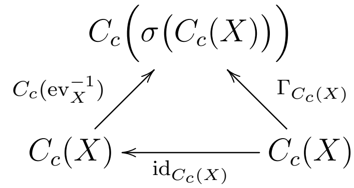
is an adjoint equivalence of categories.

Proof. It suffices to check the zig-zag identity



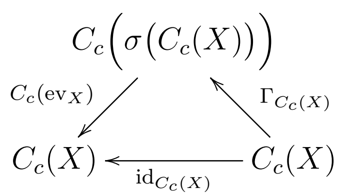
(2.28)

which translates to commutativity of the diagram[6]



(2.29)

for every locally compact Hausdorff space X. Because all morphisms here are invertible, it is equivalent to show that the diagram



(2.30)

commutes. Thus, let $\varphi \in C_c(X)$. Applying the composition along the top two arrows to this element gives

$$C_c(\mathrm{ev}_X)\big(\Gamma_{C_c(X)}(\varphi)\big) = \Gamma_{C_c(X)}(\varphi) \circ \mathrm{ev}_X \tag{2.31}$$

by Proposition 2.3. This is a map $X \to \sigma\big(C_c(X)\big) \to \mathbb{C}$ and is therefore determined pointwise. So let $x \in X$ and apply this map to it. The result is

$$\Gamma_{C_c(X)}(\varphi)\Big(\mathrm{ev}_X(x)\Big) \overset{(2.8)}{=\!=} \mathrm{ev}_X(x)\big(\varphi\big) \overset{(2.18)}{=\!=} \varphi(x), \tag{2.32}$$

which proves the proposition. ■

Theorem 2.33. *Using the same notation as above, the same functors and natural isomorphisms exhibit an adjoint equivalence of categories on* **ucC*-Alg** *and* **lcHaus**.

Proof. Essentially the same proof works. ■

Although it's unnecessary for our purposes, there are also functors

$$\mathbf{cC^*\text{-}Alg} \to \mathbf{ucC^*\text{-}Alg} \qquad \& \qquad \mathbf{lcHaus} \to \mathbf{cHaus} \tag{2.34}$$

given by adjoining $\mathbb{C}$ and taking the one-point compactification, respectively. All these functors satisfy many nice properties with respect to the functors introduced above.

[6] The arrows seem a little funny due to the contravariance of the functors.

References

[1] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, 1st ed. Studies in Advanced Mathematics, CRC Press, 1994.