From observables and states to Hilbert space and back: a 2-categorical adjunction CUNY GC Mathematical Physics, Fourier Analysis, and Applications Seminar May 13, 2016

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C^*-algebras I

Definition

A *unital Banach algebra* is a vector space $\mathcal{A}$ together with

i) a binary multiplication operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$,

ii) a norm $\| \cdot \| : \mathcal{A} \to \mathbb{R}_{\geq 0}$,

iii) and an element $1_{\mathcal{A}} \in \mathcal{A}$.

The multiplication must be distributive over vector addition, the element $1_{\mathcal{A}}$ must satisfy the condition $a1_{\mathcal{A}} = 1_{\mathcal{A}}a = a$ for all $a \in \mathcal{A}$, and finally, all Cauchy sequences must converge.

Definition

A *unital C*-algebra* is a unital Banach algebra $\mathcal{A}$ with an involution $^* : \mathcal{A} \to \mathcal{A}$ that is an anti-homomorphism for the multiplication and satisfies $\|aa^*\| = \|a\|^2$ for all $a \in \mathcal{A}$. An element $a \in \mathcal{A}$ is *self-adjoint* if $a^* = a$, an *isometry* if $a^*a = 1_{\mathcal{A}}$, and *unitary* if $a^*a = 1_{\mathcal{A}} = aa^*$.

C^*-algebras II

Definition

A *map of C*-algebras* is a (bounded) linear map $f : \mathcal{A} \to \mathcal{A}'$ from a C^*-algebra, $\mathcal{A}$, to another one , $\mathcal{A}'$, such that $f(a^*) = f(a)^*$, $f(a_1 a_2) = f(a_1)f(a_2)$, and $f(1_{\mathcal{A}}) = 1_{\mathcal{A}'}$ for all $a, a_1, a_2 \in \mathcal{A}$.

Definition

Let **C*-Alg** be the category of unital C^*-algebras, namely an object of **C*-Alg** consists of a unital C^*-algebra $\mathcal{A}$ and a morphism $f : \mathcal{A} \to \mathcal{A}'$ is a map of unital C^*-algebras.

Throughout this talk, *all* C^*-algebras will be assumed unital. We think of a C*-algebra $\mathcal{A}$ as the full algebra of observables of a physical system. An example to relate to is the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ with involution given by taking the adjoint. However, there are physical situations where there is no canonical choice of a Hilbert space (eg. different phases of matter).

States on C^*-algebras I

Definition

Given a C*-algebra $\mathcal{A}$, a *state* on $\mathcal{A}$ is a bounded linear function $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(1_{\mathcal{A}}) = 1$ and $\omega(a^* a) \geq 0$ for all $a \in \mathcal{A}$. Denote the set of states on a C*-algebra $\mathcal{A}$ by $\mathcal{S}(\mathcal{A})$.

As an example, let ψ be a unit normalized vector on a Hilbert space $\mathcal{H}$ with inner product denoted by $\langle \cdot \, , \, \cdot \rangle$. Then

$$\mathcal{B}(\mathcal{H}) \ni A \mapsto \langle \psi, A\psi \rangle$$

is a state on $\mathcal{B}(\mathcal{H})$. More generally, let ρ be a density matrix on $\mathcal{H}$. Then

$$\mathcal{B}(\mathcal{H}) \ni A \mapsto \mathrm{Tr}(\rho A)$$

is a state on $\mathcal{B}(\mathcal{H})$.

States on C^*-algebras II

Let **Set** be the category of sets and functions.

Proposition

The assignment

$$\begin{aligned} \mathbf{C^*\text{-}Alg}^{\mathrm{op}} &\xrightarrow{\mathcal{S}} \mathbf{Set} \\ \mathcal{A} &\mapsto \mathcal{S}(\mathcal{A}) \\ \left(\mathcal{A}' \xrightarrow{f} \mathcal{A}\right) &\mapsto \left(\mathcal{S}(\mathcal{A}') \xleftarrow{\mathcal{S}(f)} \mathcal{S}(\mathcal{A})\right), \end{aligned}$$

where $\mathcal{S}(f)$ is defined by

$$\mathcal{S}(\mathcal{A}) \ni \omega \mapsto \omega \circ f \in \mathcal{S}(\mathcal{A}')$$

is a functor.

The op in the superscript denotes the opposite category: objects are unchanged but all morphisms are reversed.

Representations of C^*-algebras I

Definition

Let **Rep**$(\mathcal{A})$ be the category of representations of the C*-algebra $\mathcal{A}$ on Hilbert spaces. This means the objects are pairs $(\pi, \mathcal{H})$ with $\mathcal{H}$ a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a map of C*-algebras. Morphisms $(\pi, \mathcal{H}) \to (\pi', \mathcal{H}')$ are intertwiners, i.e. bounded linear operators $L : \mathcal{H} \to \mathcal{H}'$ such that

$$L \circ \pi(a) = \pi'(a) \circ L \quad \text{for all } a \in \mathcal{A}.$$

Representations of C^*-algebras II

Let **Cat** be the category of categories and functors.

Proposition

The assignment

$$\begin{aligned} \mathbf{C^*\text{-}Alg}^{\mathrm{op}} &\xrightarrow{\mathbf{Rep}} \mathbf{Cat} \\ \mathcal{A} &\mapsto \mathbf{Rep}(\mathcal{A}) \\ \left(\mathcal{A}' \xrightarrow{f} \mathcal{A}\right) &\mapsto \left(\mathbf{Rep}(\mathcal{A}') \xleftarrow{\mathbf{Rep}(f)} \mathbf{Rep}(\mathcal{A})\right), \end{aligned} \tag{1.3}$$

is a functor. Here $\mathbf{Rep}(f)$*, sometimes written as* f^**, is the functor defined by sending a representation* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \mathcal{H})$ *to the representation* $(\pi \circ f : \mathcal{A}' \to \mathcal{B}(\mathcal{H}), \mathcal{H})$ *and by sending an intertwiner* $(\pi, \mathcal{H}) \xrightarrow{L} (\rho, \mathcal{V})$ *to the intertwiner* $(\pi \circ f, \mathcal{H}) \xrightarrow{L} (\rho \circ f, \mathcal{V})$.

Pointed representations of C^*-algebras I

Definition

A *cyclic vector* for a representation π of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a vector $\Omega \in \mathcal{H}$ such that

$$\{\pi(a)\Omega \ : \ a \in \mathcal{A}\}$$

is a dense subset in $\mathcal{H}$ (with respect to the norm induced by the inner product on $\mathcal{H}$). A representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ together with a cyclic vector Ω is called a *cyclic representation* and is written as a triple $(\pi, \mathcal{H}, \Omega)$. A representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ together with a vector (not necessarily cyclic) is called a *pointed representation*.

Pointed representations of C^*-algebras II

Definition

Let $\mathbf{Rep}^\bullet(\mathcal{A})$ be the category of pointed representations of $\mathcal{A}$. Namely, an object of $\mathbf{Rep}^\bullet(\mathcal{A})$ consists of a pointed representation $(\pi, \mathcal{H}, \Omega)$ of $\mathcal{A}$. A morphism $(\pi, \mathcal{H}, \Omega) \to (\pi', \mathcal{H}', \Omega')$ is an intertwiner $L : \mathcal{H} \to \mathcal{H}'$ of representations such that

$$L(\Omega) = \Omega' \qquad \& \qquad L^* L = \mathrm{id}_{\mathcal{H}}. \tag{1.4}$$

Let $\mathbf{Rep}^\odot(\mathcal{A})$ be the subcategory of $\mathbf{Rep}^\bullet(\mathcal{A})$ of cyclic representations.

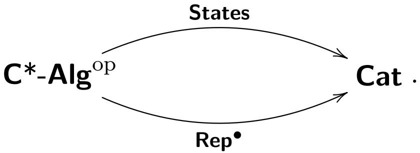
Proposition

$\mathbf{Rep}^\bullet : \mathbf{C^*\text{-}Alg}^{\mathrm{op}} \to \mathbf{Cat}$ *defined analogously to* $\mathbf{Rep}$ *is a functor.*

Warning: $\mathbf{Rep}^\odot$ is *not* a functor because if $(\pi, \mathcal{H}, \Omega)$ is a cyclic representation and $f : \mathcal{A}' \to \mathcal{A}$ is a map of C^*-algebras, then $(\pi \circ f, \mathcal{H}, \Omega)$ need *not* be cyclic!

Back to states for a moment

Let **States**$(\mathcal{A})$ denote the discrete category associated to $\mathcal{S}(\mathcal{A})$, namely all morphisms are the identities. Then **States** : **C*-Alg**$^{\mathrm{op}} \to$ **Cat** also defines a functor. Putting all this together, we have two functors



It would be nice if we could relate these two functors. A relationship between functors is described mathematically by a natural transformation.

Restriction of pure states to algebras I

Construction

Let $(\pi, \mathcal{H}, \Omega)$ *be a pointed representation of a* C^**-algebra* $\mathcal{A}$*. The vector* Ω *defines a state* ω_Ω *on* $\mathcal{B}(\mathcal{H})$ *by the formula*

$$\mathcal{B}(\mathcal{H}) \ni B \mapsto \omega_\Omega(B) := \langle \Omega, B\Omega \rangle.$$

Pulling this state back along π *defines a state* $\omega_\Omega \circ \pi : \mathcal{A} \to \mathbb{C}$ *on* $\mathcal{A}$*. This state is also denoted by* $\textbf{rest}_{\mathcal{A}}\big((\pi, \mathcal{H}, \Omega)\big)$.

Restriction of pure states to algebras II

Lemma

Let $L : (\pi, \mathcal{H}, \Omega) \to (\pi', \mathcal{H}', \Omega')$ *be a morphism of pointed representations of* $\mathcal{A}$. *Then,*

$$\omega_{\Omega'} \circ \pi' = \omega_{\Omega} \circ \pi,$$

i.e. the two states $\mathbf{rest}_{\mathcal{A}}\big((\pi, \mathcal{H}, \Omega)\big)$ *and* $\mathbf{rest}_{\mathcal{A}}\big((\pi', \mathcal{H}', \Omega')\big)$ *are equal.*

Proof.

For any $a \in \mathcal{A}$,

$$\begin{aligned} \langle \Omega', \pi'(a)\Omega' \rangle &= \langle L(\Omega), \pi'(a)L(\Omega) \rangle = \langle L(\Omega), L\pi(a)\Omega \rangle \\ &= \langle \Omega, L^* L\pi(a)\Omega \rangle = \langle \Omega, \pi(a)\Omega \rangle, \end{aligned}$$

since L is an intertwiner and $L^* L = \mathrm{id}$. ∎

Restriction of pure states to algebras III

Proposition

For every C^*-algebra $\mathcal{A}$, the assignment

$$\mathbf{Rep}^\bullet(\mathcal{A})_0 \ni (\pi, \mathcal{H}, \Omega) \mapsto \omega_\Omega \circ \pi \in \mathbf{States}(\mathcal{A})_0$$

$$\mathbf{Rep}^\bullet(\mathcal{A})_1 \ni \Big((\pi, \mathcal{H}, \Omega) \xrightarrow{L} (\pi', \mathcal{H}', \Omega') \Big) \mapsto \mathrm{id}_{\omega_\Omega \circ \pi} \in \mathbf{States}(\mathcal{A})_1$$

from our previous construction defines a functor $\mathbf{rest}_\mathcal{A} : \mathbf{Rep}^\bullet(\mathcal{A}) \to \mathbf{States}(\mathcal{A})$, *i.e. a morphism in* **Cat**.

Restriction of pure states to algebras IV

Proposition

rest, *defined by* $\mathcal{A} \mapsto \textbf{rest}_{\mathcal{A}}$, *is a natural transformation*

$$\mathbf{C^*\text{-}Alg}^{\mathrm{op}} \underset{\mathbf{Rep}^{\bullet}}{\overset{\mathbf{States}}{\rightrightarrows}} \mathbf{Cat} \quad \left(\mathbf{rest} : \mathbf{Rep}^{\bullet} \Rightarrow \mathbf{States}\right).$$

Proof.

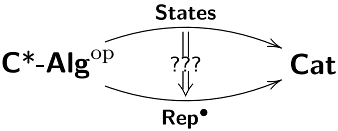
Let $f : \mathcal{A}' \to \mathcal{A}$ be a morphism of C^*-algebras. Then it is not difficult to see that the diagram

$$\begin{array}{ccc} \mathbf{Rep}^{\bullet}(\mathcal{A}) & \xrightarrow{\mathbf{rest}_{\mathcal{A}}} & \mathbf{States}(\mathcal{A}) \\ {\scriptstyle \mathbf{Rep}^{\bullet}(f)}\downarrow & & \downarrow{\scriptstyle \mathbf{States}(f)} \\ \mathbf{Rep}^{\bullet}(\mathcal{A}') & \xrightarrow[\mathbf{rest}_{\mathcal{A}'}]{} & \mathbf{States}(\mathcal{A}') \end{array}$$

commutes. ■

From states to representations?

An obvious question arises: is there a way to go back? Namely, given a state, can we construct a representation on which that state is realized as a pure state?



Asking the question this way is meaningless. Of course there are ways back. For instance, assign to every state the trivial representation. A better question to ask is if there is an "optimal" way back. The meaning of "optimal" is captured precisely by the notion of an adjunction.

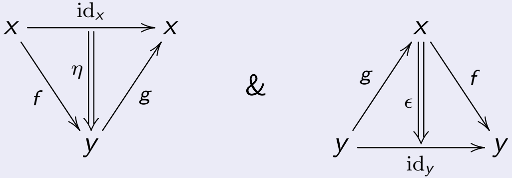
Adjunctions in 2-categories I

Definition

Let $\mathcal{C}$ be a (strict) 2-category. An *adjunction* in $\mathcal{C}$ consists of a pair of objects x, y in $\mathcal{C}$, a pair of morphisms

$$x \underset{g}{\overset{f}{\rightleftarrows}} y$$

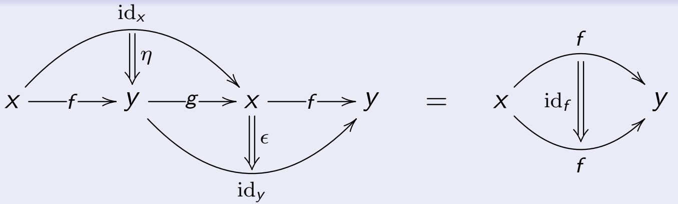
and a pair of 2-morphisms



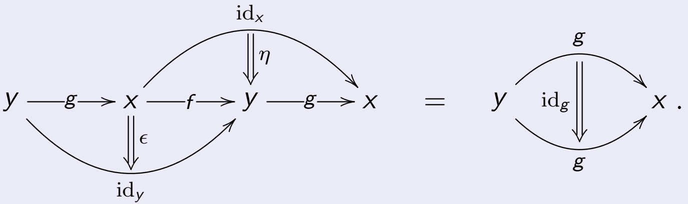
satisfying the zig-zag identities

Adjunctions in 2-categories II

Definition



and



An adjunction as above is typically written as a quadruple (f, g, η, ϵ) and we say f is *left-adjoint* to g.

Adjunctions in 2-categories III

Adjunctions abound. Here is a list of examples.[1]

- Every isomorphism $x \xrightarrow{f} y$ with inverse $y \xrightarrow{g} x$ provides two adjunctions $(f, g, \mathrm{id}, \mathrm{id})$ and $(g, f, \mathrm{id}, \mathrm{id})$.
- Let **CMon** be the category of commutative monoids and **Ab** be the category of abelian groups. Let $\mathbf{Ab} \xrightarrow{g} \mathbf{CMon}$ be the functor that forgets the inverse operation. The left adjoint of g exists and is sometimes called the Grothendieck construction. This is used, for example, to define the K-theory of a space. There is a generalization of this left adjoint to non-abelian groups. It is called the universal enveloping group functor.

[1] Many of these examples can be found online at http://math.stackexchange.com/questions/46708/a-bestiary-about-adjunctions?rq=1

Adjunctions in 2-categories IV

- Let $F : A \to B$ be a function of sets. Let $\mathcal{P}(A)$ be the category whose objects are subsets of A and whose morphisms are inclusions. Let $F^* : \mathcal{P}(B) \to \mathcal{P}(A)$ be the inverse image functor, i.e. $F^*(\beta) := \{a \in A | f(a) \in \beta\}$. Then the left adjoint of F^* is the push-forward F_*, i.e. $F_*(\alpha) := \{f(a) | a \in \alpha\}$. The right adjoint of F^* denoted by G_f is given by $G_f(\alpha) = \{b \in B | F^*(b) \subset \alpha\}$.
- Let **LieGrp** be the category of Lie groups and **LieAlg** be the category of Lie algebras. Let T_e : **LieGrp** $\to$ **LieAlg** be the tangent space at the identity functor. The left adjoint is the exponential map $\exp$: **LieAlg** $\to$ **LieGrp** taking a Lie algebra to its associated universal cover Lie group. Given a Lie group G, $\epsilon_G : \exp(T_e G) \to G$ is the universal covering map.

Adjunctions in 2-categories V

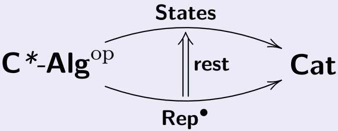
- Let G be a compact Lie group and H a closed subgroup of G. Let $i : \mathbf{Rep}(G) \to \mathbf{Rep}(H)$ be the restriction of a representation of G to a representation of H (technical assumptions can be found in Folland's QFT book). The right adjoint to i is the induced representation functor obtained by taking a representation $(\pi, \mathcal{H})$ of H, viewing G as a principal H bundle from the short exact sequence $H \to G \to G/H$, taking the associated bundle using the representation $(\pi, \mathcal{H})$, and then taking L^2-sections.

In all the above examples, I used the word "the" adjoint as if there was only a single one. This is basically true. Adjoints are unique up to canonical isomorphism.

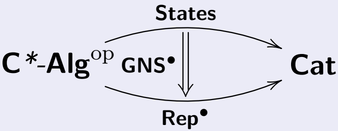
From states to representations

Theorem (AP)

The natural transformation



has a left adjoint



given by the Gelfand-Naimark-Segal (GNS) construction.

Note that this adjunction is in the 2-category $\mathrm{Fun}\big(\textbf{C*-Alg}^{\mathrm{op}}, \textbf{Cat}\big)$, where the 2-categorical structure comes from the 2-categorical structure of **Cat**. The objects of $\mathrm{Fun}\big(\textbf{C*-Alg}^{\mathrm{op}}, \textbf{Cat}\big)$ are functors.

The GNS construction for a fixed C^*-algebra I

Let $\mathcal{A}$ be a C^*-algebra. We will first define a functor

$$\mathbf{GNS}^{\bullet}_{\mathcal{A}} : \mathbf{States}(\mathcal{A}) \to \mathbf{Rep}^{\bullet}(\mathcal{A}),$$

which is determined by what it does on objects of $\mathbf{States}(\mathcal{A})$. Hence, let $\omega : \mathcal{A} \to \mathbb{C}$ be a state on $\mathcal{A}$. Then the function

$$\begin{aligned} \mathcal{A} \times \mathcal{A} &\to \mathbb{C} \\ (b, a) &\mapsto \omega(b^* a) \end{aligned}$$

is a bilinear map that is skew-conjugate in the first variable. Furthermore, it satisfies

$$\omega(b^* a) = \overline{\omega(a^* b)} \qquad \forall\ a, b \in \mathcal{A}$$

and

$$|\omega(b^* a)|^2 \leq \omega(b^* b)\omega(a^* a) \qquad \forall\ a, b \in \mathcal{A}.$$

The GNS construction for a fixed C^*-algebra II

Define the set of *null-vectors* by

$$\mathcal{N}_\omega := \{a \in \mathcal{A} \mid \omega(a^* a) = 0\}.$$

Then $\mathcal{N}_\omega$ is an ideal inside $\mathcal{A}$, meaning that $ab \in \mathcal{N}_\omega$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{N}_\omega$. Because of this, we can define the quotient

$$\mathcal{H}_\omega := \mathcal{A}/\mathcal{N}_\omega.$$

Write the equivalence class of $a \in \mathcal{A}$ as $[a]$ and define an inner product

$$\begin{aligned} \mathcal{H}_\omega \times \mathcal{H}_\omega &\xrightarrow{\langle \cdot\, ,\, \cdot \rangle_\omega} \mathcal{H}_\omega \\ ([b],[a]) &\mapsto \omega(b^* a) \end{aligned}$$

by choosing lifts of the equivalence classes. Complete $\mathcal{H}_\omega$ with respect to this inner product. We use the same notation $\mathcal{H}_\omega$ for this completion.

The GNS construction for a fixed C^*-algebra III

Let $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the representation defined by

$$\pi_\omega(a)[b] := [ab]$$

for all $a \in \mathcal{A}$ and $[b] \in \mathcal{H}_\omega$. One should check this is indeed well-defined. Furthermore, let $\Omega_\omega := [1_{\mathcal{A}}]$. Using all this, we define

$$\mathbf{GNS}^{\bullet}_{\mathcal{A}}(\omega) := (\pi_\omega, \mathcal{H}_\omega, \Omega_\omega).$$

The GNS construction for a morphism of C^*-algebras I

Now let $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ be a morphism of C^*-algebras and $\omega : \mathcal{A} \to \mathbb{C}$ a state on $\mathcal{A}$. Then f induces a state on $\mathcal{A}'$ via pullback $\omega \circ f : \mathcal{A}' \to \mathbb{C}$. By applying **GNS$^\bullet$**, we get two pointed representations $(\pi_{\omega\circ f}, \mathcal{H}_{\omega\circ f}, [1_{\mathcal{A}'}])$ and $(\pi_\omega, \mathcal{H}_\omega, [1_{\mathcal{A}}])$, the first on $\mathcal{A}'$ and the latter on $\mathcal{A}$. There is a canonical map $L_f : \mathcal{H}_{\omega\circ f} \to \mathcal{H}_\omega$ obtained from the diagram

$$
\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{\quad f \quad} & \mathcal{A} \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}'/\mathcal{N}_{\omega\circ f} = \mathcal{H}_{\omega\circ f} & \overset{L_f}{\dashrightarrow} & \mathcal{H}_\omega = \mathcal{A}/\mathcal{N}_\omega
\end{array}
$$

given by

$$L_f([a']) := [f(a')]$$

for all $[a'] \in \mathcal{H}_{\omega\circ f}$.

The GNS construction for a morphism of C^*-algebras II

L_f is well-defined and is an intertwiner $(\pi_{\omega\circ f}, \mathcal{H}_{\omega\circ f}, [1_{\mathcal{A}'}]) \to (\pi_\omega \circ f, \mathcal{H}_\omega, [1_{\mathcal{A}}])$ of pointed representations of $\mathcal{A}'$, which means that L_f is an isometry and that the diagram

$$\begin{array}{ccc} \mathcal{H}_{\omega\circ f} & \xrightarrow{L_f} & \mathcal{H}_\omega \\ {\scriptstyle \pi_{\omega\circ f}(a')}\downarrow & & \downarrow{\scriptstyle \pi_\omega\big(f(a')\big)} \\ \mathcal{H}_{\omega\circ f} & \xrightarrow[L_f]{} & \mathcal{H}_\omega \end{array}$$

commutes for all $a' \in \mathcal{A}'$. Note also that $L_f([1_{\mathcal{A}'}]) = [f(1_{\mathcal{A}'})] = [1_{\mathcal{A}}]$ since f is a (unital) C^*-algebra morphism.

The GNS construction for a morphism of C^*-algebras III

Hence, although the diagram

$$\begin{array}{ccc} \mathbf{States}(\mathcal{A}) & \xrightarrow{\mathbf{GNS}^\bullet_{\mathcal{A}}} & \mathbf{Rep}^\bullet(\mathcal{A}) \\ {\scriptstyle \mathbf{States}(f)}\downarrow & & \downarrow{\scriptstyle \mathbf{Rep}^\bullet(f)} \\ \mathbf{States}(\mathcal{A}') & \xrightarrow[\mathbf{GNS}^\bullet_{\mathcal{A}'}]{} & \mathbf{Rep}^\bullet(\mathcal{A}') \end{array}$$

does not commute, there is a natural transformation $\mathbf{GNS}^\bullet_f$ that fits into this square

$$\begin{array}{ccc} \mathbf{States}(\mathcal{A}) & \xrightarrow{\mathbf{GNS}^\bullet_{\mathcal{A}}} & \mathbf{Rep}^\bullet(\mathcal{A}) \\ {\scriptstyle \mathbf{States}(f)}\downarrow & \overset{\mathbf{GNS}^\bullet_f}{\Nearrow} & \downarrow{\scriptstyle \mathbf{Rep}^\bullet(f)} \\ \mathbf{States}(\mathcal{A}') & \xrightarrow[\mathbf{GNS}^\bullet_{\mathcal{A}'}]{} & \mathbf{Rep}^\bullet(\mathcal{A}') \end{array}$$

defined on a state ω by

$$\mathbf{GNS}^\bullet_f(\omega) := L_f.$$

The leftovers of the GNS adjunction

The rest of the construction is not difficult. One has to construct the 2-morphisms of the adjunction and verify the axioms of an adjunction.

So... what's the significance of this perspective? I'm honestly not sure. I just wanted to know if there was a categorical description of the GNS construction, and I believe this description as an adjunction is concise and captures the essential features. In particular, the adjunction property implies that $\mathbf{GNS}^{\bullet}_{\mathcal{A}}(\omega)$ is the "smallest" representation on which the state ω can be described as a pure state.