I) Preliminaries

A Quantum system is specified by a Hilbert space $\mathcal{H}$ together with a self-adjoint operator $\hat{H}$ called the Hamiltonian. States are defined with respect to what we measure.

Recall [Hall], a state (in quantum mechanics) consists of a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ (thought of as a family of expectation values) such that

1) $\Phi(\mathrm{id}) = 1$ (needed for probabilistic interpretation)
2) $\Phi(A) \in \mathbb{R}$ if $A^{\dagger} = A$
3) $\Phi(A) \geq 0$ if $A^{\dagger} = A$ and A is nonnegative
4) For any sequence $\{A_n\}$ in $\mathcal{B}(\mathcal{H})$, if $\lim_{n\to\infty} \|A_n\psi - A\psi\| = 0$ for all $\psi \in \mathcal{H}$ then $\lim_{n\to\infty} \Phi(A_n)$

This definition is more reasonable from a physical point of view but is not the one usually introduced in physics courses. Perhaps a more common defin is given by density matrices. A density matrix is an operator $\rho \in \mathcal{B}(\mathcal{H})$ that is self-adjoint, non-negative, and $\mathrm{trace}(\rho) = 1$.

Theorem $\{\text{density matrices } \rho\} \xrightarrow{\sim} \{\text{states}\}$ defined by $\rho \mapsto \Phi_\rho$ with $\Phi_\rho(A) := \mathrm{trace}(\rho A)$ is an isomorphism. See Hall ch 19 for the proof.

The <u>time evolution</u> of a state ρ with respect to a Hamiltonian $\hat{H}$ is governed by the IVP

$$(*) \quad \frac{d\rho}{dt} = -\frac{1}{i\hbar}[\rho, \hat{H}] \qquad \rho(t=0) = \rho_0$$

In many situations, pure states suffice to describe a system. A state ρ is <u>pure</u> if $\rho = P_{\ell_\psi}$ where P_{ℓ_ψ} is the projection operator onto the span ℓ_ψ of ψ. It is common to write this as $|\psi\rangle\langle\psi|$. This is consistent with the usual Schrödinger equation since the solution to $(*)$ is

$$\rho(t) = e^{-\frac{it}{\hbar}\hat{H}} \rho_0 e^{\frac{it}{\hbar}\hat{H}}$$

and so if $\rho_0 = |\psi_0\rangle\langle\psi_0|$ then

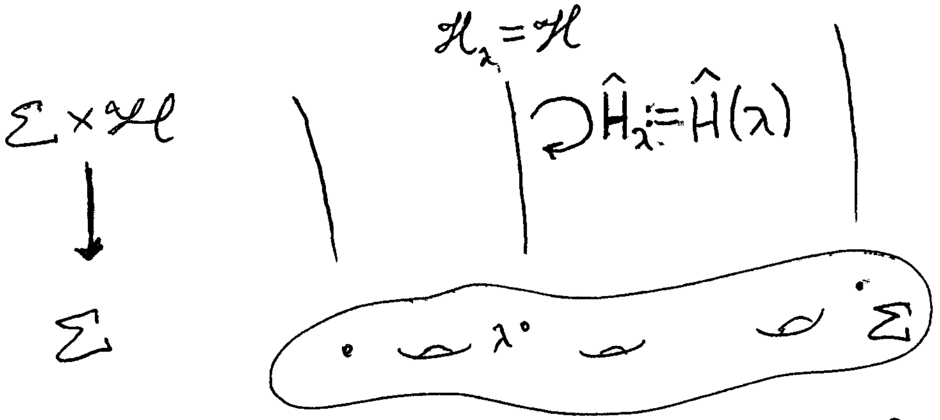
$$e^{-\frac{it}{\hbar}\hat{H}} |\psi_0\rangle\langle\psi_0| e^{\frac{it}{\hbar}\hat{H}} = |e^{-\frac{it\hat{H}}{\hbar}}\psi_0\rangle\langle e^{-\frac{it\hat{H}}{\hbar}}\psi_0|$$

$$= |\psi(t)\rangle\langle\psi(t)|$$

II) <u>Families of quantum systems</u>

Occasionally, a quantum system (as above) may depend on some parameter that can be controlled (say, by the experimenter) or sometimes it can't be controlled. For example, a Hamiltonian might depend on a magnetic field $\vec{B}$. Another example occurs in the context of non-interacting electrons in an infinite crystal. There, the parameter space is called the Brillouin zone.

A *family of Hamiltonians* on $\mathcal{H}$ parametrized by Σ consists of a map $\hat{H} : \Sigma \longrightarrow \mathcal{B}_{s.a.}(\mathcal{H})$.
Frequently, we want additional assumptions on $\hat{H}$ as well as additional structure on Σ.
For example, Σ might be a measure space and/or a smooth (or topological) space.
It is convenient to think of $\hat{H}$ as acting fiberwise on a copy of $\mathcal{H}$ over each point $\lambda \in \Sigma$.



The reason for why this is convenient will be explained later on in the discussion of the Berry phase. A nice family of Hamiltonians will decompose the Hilbert space into (eigen) subspaces suitable for analyzing how the Hamiltonian acts on the system. In general, these subspaces will come from non-trivial
12 sub-bundles of the trivial bundle $\Sigma \times \mathcal{H} \to \Sigma$.

III) Vector bundles with connection

<u>Defn</u> A *complex vector bundle* over Σ with fiber V consists of a space E and a map $p : E \to \Sigma$ such that $p^{-1}(\lambda) =: E_\lambda$ is a $\mathbb{C}$-vector space and for every $\lambda \in \Sigma$ there exists an open neighborhood U_λ around λ together

with an isomorphism $\phi_\lambda : E|_{\mathcal{U}_\lambda} \longrightarrow \mathcal{U}_\lambda \times V$ (here $E|_{\mathcal{U}_\lambda} := p^{-1}(\mathcal{U}_\lambda)$) that is linear on each fiber and such that

$$\begin{tikzcd} E|_{\mathcal{U}_\lambda} \arrow[rr, "\phi_\lambda"] \arrow[dr, "p"'] & & \mathcal{U}_\lambda \times V \arrow[dl, "\pi"] \\ & \mathcal{U}_\lambda & \end{tikzcd}$$

commutes.

E is called the <u>total space</u> and Σ the <u>base space</u>. The <u>rank</u> of E is the dimension of V.

<u>Def'n</u> Let $p : E \to \Sigma$ be a vector bundle. A <u>section</u> of p is a map $s : \Sigma \to E$ such that

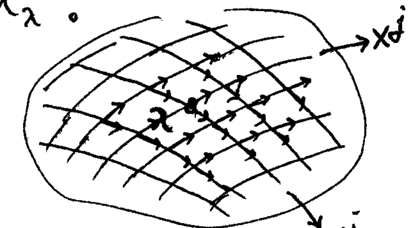
$$\begin{tikzcd} & E \arrow[dr, "p"] & \\ \Sigma \arrow[ur, "s"] \arrow[rr, "id"'] & & \Sigma \end{tikzcd}$$

commutes.

The set of sections is denoted by $\Gamma(p)$ or $\Gamma(E)$.

<u>Example</u> (the tangent bundle)

Around every point λ in a manifold Σ there are coordinates $\{x^i\}$ in a neighborhood $\mathcal{U}_\lambda$.



These coordinates can be used to construct the <u>vector fields</u> $\left\{\frac{\partial}{\partial x^i} =: \partial_i\right\}$ (vector fields are linear maps sending functions to functions satisfying the Leibniz rule). The vector fields restricted to each point of $\mathcal{U}_\lambda$ span a vector space called the tangent space (drawn in black above). One uses changes of coordinates to glue together the total space of a bundle called the tangent bundle $T\Sigma \to \Sigma$.

A section of the tangent bundle is the same thing as a vector field. The set of vector fields is written as $\mathfrak{X}(\Sigma)$

Given a section of an arbitrary vector bundle, one would like to take its derivative along vector fields. This is made possible with the choice of a covariant derivative operator.

Defn Let $p : E \to \Sigma$ be a vector bundle and $X \in \mathfrak{X}(\Sigma)$. The covariant derivative w.r.t. X is a function

$\nabla_X : \Gamma(E) \to \Gamma(E)$ satisfying

i) $\nabla_X(\alpha s) = \alpha \nabla_X(s) \quad \forall \alpha \in \mathbb{C}, \ s \in \Gamma(E)$

ii) $\nabla_X(s+t) = \nabla_X(s) + \nabla_X(t) \quad \forall s,t \in \Gamma(E)$

iii) $\nabla_{fX}(s) = f\nabla_X(s) \quad \forall f \in C^\infty(\Sigma;\mathbb{R}), \ s \in \Gamma(E)$

iv) $\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s) \quad \forall X,Y \in \mathfrak{X}(\Sigma)$

v) $\nabla_X(fs) = X(f)\cdot s + f\nabla_X(s) \quad \forall f \in C^\infty(\Sigma$

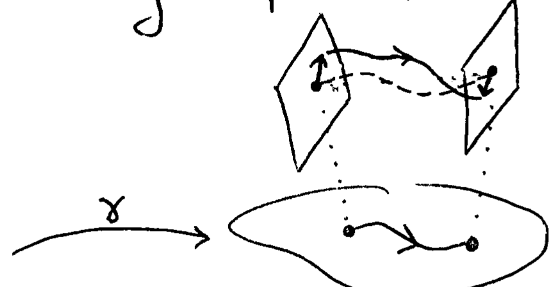
∇ is said to be a connection on $p : E \to \Sigma$.

Example The trivial connection on $\Sigma \times \mathcal{H} \to \Sigma$ is given by

$$d_X(s) \equiv \nabla_X(s) := \sum_n X(s_n)e_n$$

where $\{e_n\}$ is a basis and $s = \sum_n s_n e_n$.

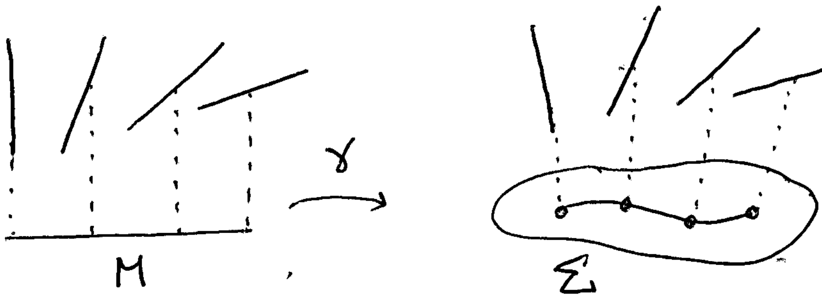
Connections satisfy a unique path lifting property (similar to covering spaces)



To precisely state this in terms of formulas and differential equations we describe pullback
27 bundles and connections.

If the fibers are Hilbert spaces $\mathcal{H}_\lambda$ then ∇ is Hermitian if $X(\langle s,t\rangle) = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$.

<u>Defn</u> Let $M \xrightarrow{\gamma} \Sigma$ be a map and $p: E \to \Sigma$ a bundle. Then the <u>pullback bundle</u> of p along γ is given by $\gamma^* E := \{ (x,v) \in M \times E \mid \gamma(x) = p(v) \}$ with the projection map $\gamma^* E \longrightarrow M$ being the obvious one.
Note that the other projection is an isomorphism of fibers. Thus, we can draw this as



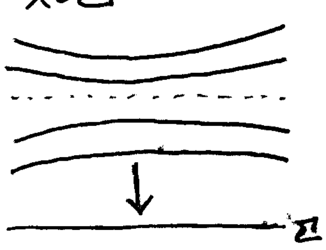
If $p: E \longrightarrow \Sigma$ has a connection ∇ then
$\left((\gamma^* \nabla)_X (s) \right)(x) := \tilde{\gamma}_x^{-1} \left(\left(\nabla_{X_x} (s) \right)(\gamma(x)) \right)$, where $x \in M$, $X \in \mathfrak{X}(M)$, $X_x = X(x) \in T_x M$, $s \in \Gamma(\gamma^* E)$, and $\tilde{\gamma}_x^{-1} : \gamma^* E_x \to E_{\gamma(x)}$ is the isomorphism mentioned above, defines a connection on $\gamma^* E$. This is the <u>pullback connection</u>.

<u>Defn</u> Using the notation above but setting $M = [0,1]$, and $\gamma(0) = x$ & $\gamma(1) = y$, and $v \in E_x$, the <u>parallel transport</u> of v along γ at y is the value $\tilde{\gamma}(\psi(1)) \in E_y$ of the (unique) sol'n to the IVP

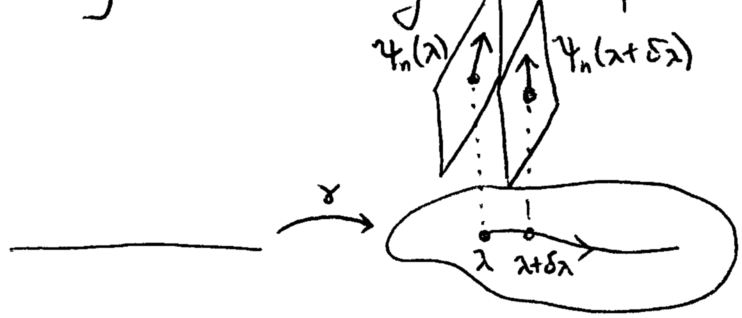
$$(\gamma^* \nabla)_{\frac{d}{dt}} \psi = 0 \qquad \psi(0) = \tilde{\gamma}_x^{-1}(v) \qquad 34$$

IV) <u>The Berry connection</u>

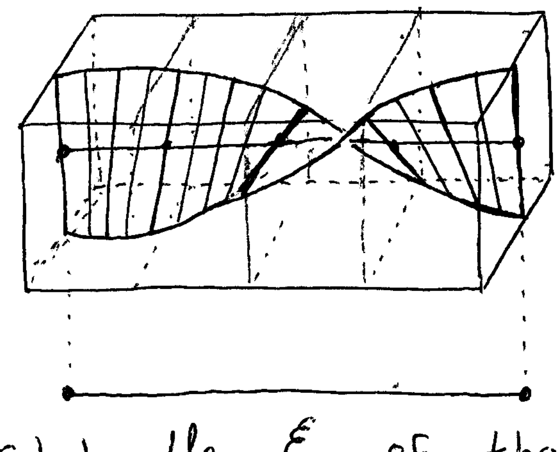
Given a sufficiently nice family of Hamiltonians $\hat{H}: \Sigma \longrightarrow \mathcal{B}_{s.a.}(\mathcal{H})$, the union of the spectra $\bigcup_{\lambda \in \Sigma} \sigma(\hat{H}_\lambda)$ is covering space over Σ. Under these conditions, $\hat{H}$ is said to be an <u>adiabatic family</u> of Hamiltonians.



Physically, one imagines beginning with an eigenstate $\Psi_n(\lambda) \in \mathcal{H}$ of $\hat{H}_\lambda$ and perturbing it along a path $[0,1] \xrightarrow{\gamma} \Sigma$



so that $\psi_n(\lambda+\delta\lambda)$ is an eigenstate of $\hat{H}_{\lambda+\delta\lambda}$ (here "$\lambda+\delta x$" is just a symbol representing a point close to λ — we do not need to assume any kind of addition structure on Σ). The previous definition makes this intuitive picture precise. Given such an adiabatic family of Hamiltonians, if we let $\mathcal{E}_n(\lambda)$ be the n^{th} eigenspace of $\hat{H}_\lambda$ with dimension $d_n(\lambda)$, then $\mathcal{E}_n := \bigcup_{\lambda\in\Sigma} \mathcal{E}_n(\lambda)$ defines a rank d_n sub-bundle of the trivial bundle $\Sigma \times \mathcal{H} \to \Sigma$. Such sub-bundles can be nontrivial!

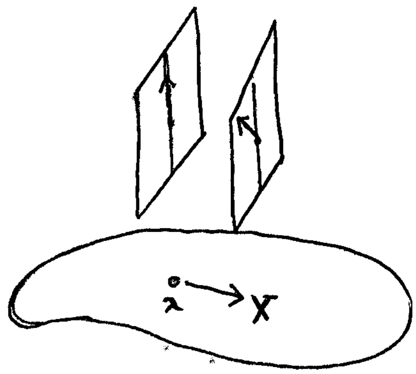


$S^1 = \Sigma$ (identify ends)

<u>Prop</u> Given any sub-bundle $\mathcal{E}$ of the trivial bundle $\Sigma\times\mathcal{H}\to\Sigma$, there is a projection operator $P : \Sigma \longrightarrow \mathcal{B}_{s.a}(\mathcal{H})$ such that $\text{Image}(P_\lambda) = \mathcal{E}_\lambda$. Let $\mathcal{E} \xhookrightarrow{i} \Sigma\times\mathcal{H}$ be the inclusion and let d be the trivial connection on $\Sigma\times\mathcal{H}\to\Sigma$. Then $P\circ d\circ i$, defined by $(P\circ d\circ i)_X(s) := P(X(i(s)))$, is a connection on $\mathcal{E}$.

<u>Defn</u> When $\mathcal{E} = \mathcal{E}_n$, the n^{th} eigen subbundle associated to an adiabatic family of Hamiltonians, $P_n\circ d\circ i_n$ is called the <u>Berry connection</u>.

- This formula is very intuitive:



treat the section of $\mathcal{E}$ as a section of $\Sigma \times \mathcal{H}$, take its derivative, and then project back onto $\mathcal{E}$.

<u>Special case</u> Suppose $d_n = 1$ i.e. $\mathcal{E}_n$ has rank 1. Let $\mathcal{U}$ be a coordinate chart for Σ (locally) with coordinates $\{x^i\}$. Let $\psi_n : \mathcal{U} \to \mathcal{E}_n$ be a section that does not vanish. This is occasionally written as $|\psi_n\rangle$. Then $P_n = |\psi_n\rangle\langle\psi_n|$. Let $s : \mathcal{U} \to \mathcal{E}_n$ be any other section (it could vanish for instance). Then $\exists f \in C^\infty(\mathcal{U}, \mathbb{C})$ such that $s = f\psi_n$. Using this our formula becomes

$$(P_n \circ d \circ i)_{\frac{\partial}{\partial x^j}}(s) = \left(|\psi_n\rangle\langle\psi_n| \circ d \circ i\right)_{\frac{\partial}{\partial x^j}}\left(f|\psi_n\rangle\right)$$

$$= (\partial_j f)|\psi_n\rangle + f\left(|\psi_n\rangle\langle\psi_n| \circ d \circ i\right)_{\frac{\partial}{\partial x^j}}\left(|\psi_n\rangle\right)$$

$$= (\partial_j f)|\psi_n\rangle + f\,|\psi_n\rangle\langle\psi_n|\partial_j|\psi_n\rangle$$

$$= \left(\partial_j f + f(A_n)_j\right)|\psi_n\rangle, \quad \text{where}$$

$(A_n)_j\, dx^j := \langle\psi_n|\partial_j|\psi_n\rangle dx^j$ is what most people call the Berry connection. However, this is awkward as it depends on many choices. One can show that if one chooses ψ_n' instead of ψ_n, then there exists a function $\lambda : \mathcal{U} \to U(1)$ such that $(A_n')_j = (A_n)_j + \lambda^* \partial_j \lambda$.