Notes on 2-categories

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December 30, 2015

Abstract

In this expository note, we define 2-categories, functors, pseudo-natural transformations, modifications, and all of their compositions. We also define equivalences and the many levels of inverses for all compositions. In particular, we spell out what an equivalence between 2-categories is and what a pseudonatural equivalence between 2-functors is. We prove many (though not all) statements in order to be somewhat self-contained. Absolutely none of this is novel and is merely meant as a reference needed for future works (in particular, no examples are given). This was initially the appendix to a paper but was removed in the published version.

The following is a table of some basic definitions along with their locations.

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To set some notation, the pullback of two morphisms $f : X \to Z$ and $g : Y \to Z$ will be written as

$$\begin{array}{ccc} X {}_f\!\times_g Y & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array} . \qquad (1)$$

Definition 2. A *(small) 2-category* $\mathcal{C}$ consists of the following data:

i) a set C_0, elements of which are called *objects*,

ii) a set C_1, elements of which are called *1-morphisms*,

iii) a set C_2, elements of which are called *2-morphisms*,

iv) functions

$$C_2 \underset{t}{\overset{s}{\rightrightarrows}}\!\!\overset{i}{\leftarrow} C_1 \underset{t}{\overset{s}{\rightrightarrows}}\!\!\overset{i}{\leftarrow} C_0 , \qquad (3)$$

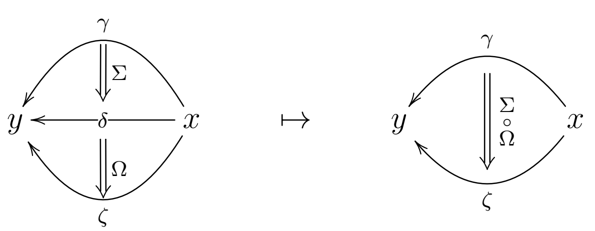
where s, t, and i stand for *source*, *target*, and *identity-assignment*, respectively,

v) functions

$$C_1 \,{}_s\!\times_t C_1 \to C_1, \qquad C_2 \,{}_s\!\times_t C_2 \to C_2, \qquad C_2 \,{}_{ss}\!\times_{tt} C_2 \to C_2 \qquad (4)$$

called ordinary composition of 1-morphisms, *vertical composition* of 2-morphisms, and *horizontal composition* of 2-morphisms, and drawn as[1]

$$z \xleftarrow{\alpha} y \xleftarrow{\beta} x \qquad \mapsto \qquad z \xleftarrow{\alpha\circ\beta} x , \qquad (5)$$



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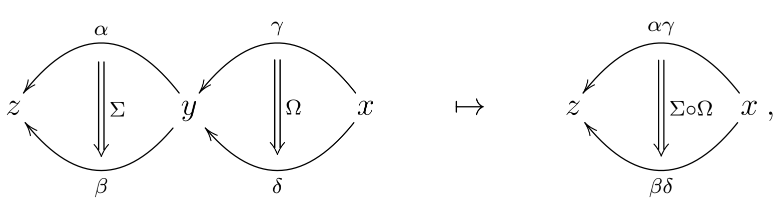
and



(7)

respectively,

[1]These drawings place restrictions on the above mentioned functions.

vi) for every triple (α, β, γ) of composable 1-morphisms, a 2-morphism

$$\begin{array}{c} (\alpha \circ \beta) \circ \gamma \\ \Big\Downarrow{\scriptstyle a_{\alpha,\beta,\gamma}} \\ \alpha \circ (\beta \circ \gamma) \end{array} \tag{8}$$

called the *associator*,

vii) and finally, for every morphism $y \xleftarrow{\alpha} x$, two 2-morphisms

$$\begin{array}{c} \alpha \circ \mathrm{id}_x \\ \Big\Downarrow{\scriptstyle l_\alpha} \\ \alpha \end{array} \qquad \& \qquad \begin{array}{c} \mathrm{id}_y \circ \alpha \\ \Big\Downarrow{\scriptstyle r_\alpha} \\ \alpha \end{array} \tag{9}$$

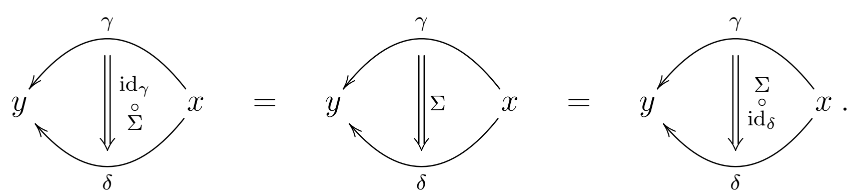
called the *left* and *right unifiers*, respectively. Here we write id_x instead of $i(x)$.

These data must satisfy the following conditions.

(a) The functions s, t, and i have to satisfy the following equalities

$$s \circ i = \mathrm{id}_{C_0} = t \circ i, \quad s \circ i = \mathrm{id}_{C_1} = t \circ i, \quad s \circ s = s \circ t, \quad \& \quad t \circ s = t \circ t. \tag{10}$$

(b) Vertical composition is associative and the identity-assigning map gives units with respect to this composition. The latter is drawn as



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(c) For every quadruple $(\alpha, \beta, \gamma, \delta)$ of composable 1-morphisms, the diagram

$$\begin{array}{ccccc} & & ((\alpha \circ \beta) \circ \gamma) \circ \delta & & \\ & \swarrow{\scriptstyle a_{\alpha\circ\beta,\gamma,\delta}} & & \searrow{\scriptstyle a_{\alpha,\beta,\gamma}\circ \mathrm{id}_\delta} & \\ (\alpha \circ \beta) \circ (\gamma \circ \delta) & & & & (\alpha \circ (\beta \circ \gamma)) \circ \delta \\ & \searrow{\scriptstyle a_{\alpha,\beta,\gamma\circ\delta}} & & \swarrow{\scriptstyle a_{\alpha,\beta\circ\gamma,\delta}} & \\ & \alpha \circ (\beta \circ (\gamma \circ \delta)) & \xLeftarrow[\mathrm{id}_\alpha \circ a_{\beta,\gamma,\delta}]{} & \alpha \circ ((\beta \circ \gamma) \circ \delta) & \end{array} \tag{12}$$

commutes. This is called the pentagon axiom.

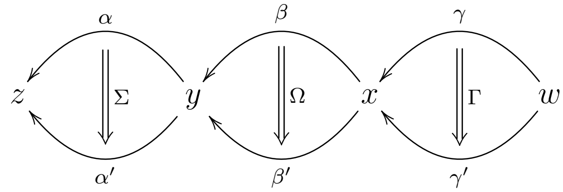
(d) For every pair $(z \xleftarrow{\alpha} y, y \xleftarrow{\beta} x)$ of composable 1-morphisms, the diagram

$$\begin{array}{ccc} \alpha \circ (\mathrm{id}_y \circ \beta) & \xLeftarrow{a_{\alpha,\mathrm{id}_y,\beta}} & (\alpha \circ \mathrm{id}_y) \circ \beta \\ & {\scriptstyle \mathrm{id}_\alpha \circ l_\beta}\searrow \quad \swarrow{\scriptstyle r_\alpha \circ \mathrm{id}_\beta} & \\ & \alpha \circ \beta & \end{array} \tag{13}$$

commutes. Furthermore,

$$\mathrm{id}_\alpha \circ \mathrm{id}_\beta = \mathrm{id}_{\alpha\circ\beta}. \tag{14}$$

(e) For every triple



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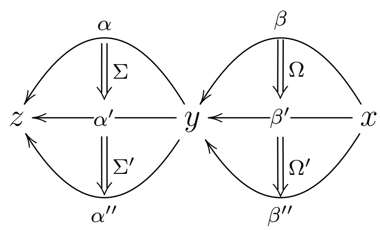
of horizontally composable 2-morphisms, the diagram

$$\begin{array}{ccc} (\alpha' \circ \beta') \circ \gamma' & \xLeftarrow{(\Sigma\circ\Omega)\circ\Gamma} & (\alpha \circ \beta) \circ \gamma \\ \Big\Downarrow a_{\alpha',\beta',\gamma'} & & \Big\Downarrow a_{\alpha,\beta,\gamma} \\ \alpha' \circ (\beta' \circ \gamma') & \xLeftarrow[\Sigma\circ(\Omega\circ\Gamma)]{} & \alpha \circ (\beta \circ \gamma) \end{array} \tag{16}$$

commutes.

(f) For every quadruple



(17)

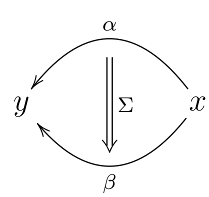
of 2-morphisms composable in the fashion indicated above,

$$\begin{matrix} (\Sigma \circ \Omega) \\ \circ \\ (\Sigma' \circ \Omega') \end{matrix} \quad = \quad \begin{pmatrix} \Sigma \\ \circ \\ \Sigma' \end{pmatrix} \circ \begin{pmatrix} \Omega \\ \circ \\ \Omega' \end{pmatrix}. \tag{18}$$

This is called the <u>*interchange law*</u>. Because of this, it is common to simply write this composition as

$$\begin{matrix} \Sigma\,\Omega \\ \Sigma'\Omega' \end{matrix} . \tag{19}$$

(g) For every 2-morphism

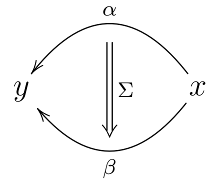


(20)

the 2-morphisms $\mathrm{id}_{\mathrm{id}_x}$ and $\mathrm{id}_{\mathrm{id}_y}$ act as right and left identities, respectively, i.e.

$$\Sigma \circ \mathrm{id}_{\mathrm{id}_x} = \Sigma \quad \& \quad \mathrm{id}_{\mathrm{id}_y} \circ \Sigma = \Sigma. \tag{21}$$

(h) All associators and unifiers are vertically invertible 2-morphisms in the following sense. A 2-morphism



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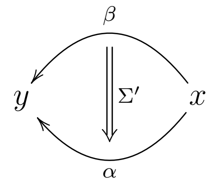
is said to be *vertically invertible* if there exists a 2-morphism



(23)

such that

$$\begin{array}{c}\Sigma\\ \circ\\ \Sigma'\end{array} = \mathrm{id}_\alpha \quad \& \quad \mathrm{id}_\beta = \begin{array}{c}\Sigma'\\ \circ\\ \Sigma\end{array}. \tag{24}$$

We shall write Σ^{-1_v} for the vertical inverse of Σ.

Definition 25. A *strict 2-category* is a 2-category whose associators and unifiers are all identity 2-morphisms.

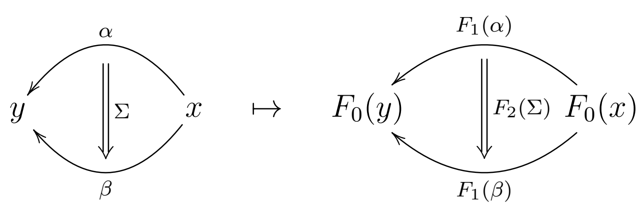
Definition 26. A *strict 2-groupoid* is a strict 2-category where all 1-morphisms and 2-morphisms are invertible.

Definition 27. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A *2-functor* F from $\mathcal{C}$ to $\mathcal{D}$, written as $F : \mathcal{C} \to \mathcal{D}$, consists of

i) functions

$$F_i : C_i \to D_i \tag{28}$$

for $i = 0, 1, 2$, that assign objects, 1-morphisms, and 2-morphisms in the following manner



(29)

ii) for every pair (α, β) of 1-morphisms in $\mathcal{C}$, a 2-morphism

$$\begin{array}{c} F_1(\alpha) \circ F_1(\beta) \\ \Downarrow{\scriptstyle c_{\alpha,\beta}} \\ F_1(\alpha \circ \beta) \end{array} \tag{30}$$

called the *compositor*,

iii) and for every object x in $\mathcal{C}$, a 2-morphism

$$\begin{array}{c} F_1(\mathrm{id}_x) \\ \Big\Downarrow {\scriptstyle u_x} \\ \mathrm{id}_{F_0(x)} \end{array} \tag{31}$$

called the <u>*unitor*</u>.

These data must satisfy the following conditions.[2]

(a) For every triple (α, β, γ) of composable 1-morphisms in $\mathcal{C}$, the diagram

$$\begin{array}{ccccc} F\Big((\alpha \circ \beta) \circ \gamma\Big) & \xLeftarrow{c_{\alpha\circ\beta,\gamma}} & F(\alpha \circ \beta) \circ F(\gamma) & \xLeftarrow{c_{\alpha,\beta}\circ \mathrm{id}_{F(\gamma)}} & \Big(F(\alpha) \circ F(\beta)\Big) \circ F(\gamma) \\ \Big\Downarrow {\scriptstyle F(a_{\alpha,\beta,\gamma})} & & & & {\scriptstyle a_{F(\alpha),F(\beta),F(\gamma)}} \Big\Downarrow \\ F\Big(\alpha \circ (\beta \circ \gamma)\Big) & \xLeftarrow[c_{\alpha,\beta\circ\gamma}]{} & F(\alpha) \circ F(\beta \circ \gamma) & \xLeftarrow[\mathrm{id}_{F(\alpha)}\circ c_{\beta,\gamma}]{} & F(\alpha) \circ \Big(F(\beta) \circ F(\gamma)\Big) \end{array} \tag{32}$$

commutes.

(b) For every 1-morphism $y \xleftarrow{\alpha} x$ in $\mathcal{C}$, the diagrams

$$\begin{array}{ccc} \mathrm{id}_{F(y)} \circ F(\alpha) & \xLeftarrow{u_y \circ \mathrm{id}_{F(\alpha)}} & F(\mathrm{id}_y) \circ F(\alpha) \\ {\scriptstyle l_{F(\alpha)}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle c_{\mathrm{id}_y,\alpha}} \\ F(\alpha) & \xLeftarrow[F(l_\alpha)]{} & F(\mathrm{id}_y \circ \alpha) \end{array} \quad \& \quad \begin{array}{ccc} F(\alpha) \circ \mathrm{id}_{F(x)} & \xLeftarrow{\mathrm{id}_{F(\alpha)} \circ u_x} & F(\alpha) \circ F(\mathrm{id}_x) \\ {\scriptstyle r_{F(\alpha)}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle c_{\alpha,\mathrm{id}_x}} \\ F(\alpha) & \xLeftarrow[F(r_\alpha)]{} & F(\alpha \circ \mathrm{id}_x) \end{array} \tag{33}$$

both commute.

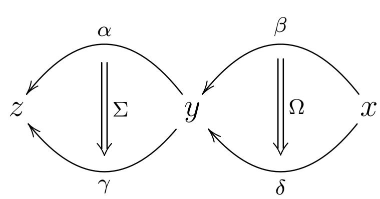
(c) For every pair (Σ, Ω) of vertically composable 2-morphisms in $\mathcal{C}$

$$F\begin{pmatrix} \Sigma \\ \circ \\ \Omega \end{pmatrix} = \begin{array}{c} F(\Sigma) \\ \circ \\ F(\Omega) \end{array}. \tag{34}$$

(d) For every 1-morphism α in $\mathcal{C}$,

$$F(\mathrm{id}_\alpha) = \mathrm{id}_{F(\alpha)}. \tag{35}$$

(e) For every pair



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[2] Just as in ordinary category theory we now write F instead of F_0, F_1, or F_2 since it will be clear from the context which one is used depending on the input.

of horizontally composable 2-morphisms in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(\gamma)\circ F(\delta) & \xLeftarrow{F(\Sigma)\circ F(\Omega)} & F(\alpha)\circ F(\beta) \\ {\scriptstyle c_{\gamma,\delta}}\Downarrow & & \Downarrow{\scriptstyle c_{\alpha,\beta}} \\ F(\gamma\circ\delta) & \xLeftarrow[F(\Sigma\circ\Omega)]{} & F(\alpha\circ\beta) \end{array} \tag{37}$$

commutes.

Definition 38. Let $\mathcal{C},\mathcal{D}$, and $\mathcal{E}$ be 2-categories and let $F:\mathcal{D}\to\mathcal{E}$ and $G:\mathcal{C}\to\mathcal{D}$ be two 2-functors. The *composition* of F and G, written as $F\circ G:\mathcal{C}\to\mathcal{E}$, is the 2-functor defined as follows.

i) The functions $(F\circ G)_i : C_i \to E_i$ are defined to be

$$(F\circ G)_i := F_i\circ G_i. \tag{39}$$

ii) For every pair (α,β) of composable 1-morphisms in $\mathcal{C}$, the compositor $c^{F\circ G}_{\alpha,\beta}$ is defined to be the vertical composite of the 2-morphisms

$$\begin{array}{c} (F\circ G)(\alpha)\circ(F\circ G)(\beta) \\ \Downarrow{\scriptstyle c^F_{G(\alpha),G(\beta)}} \\ F\Big(G(\alpha)\circ G(\beta)\Big) \\ \Downarrow{\scriptstyle F\left(c^G_{\alpha,\beta}\right)} \\ (F\circ G)(\alpha\circ\beta) \end{array} \quad , \tag{40}$$

where superscripts are used to distinguish the compositors for the two 2-functors.

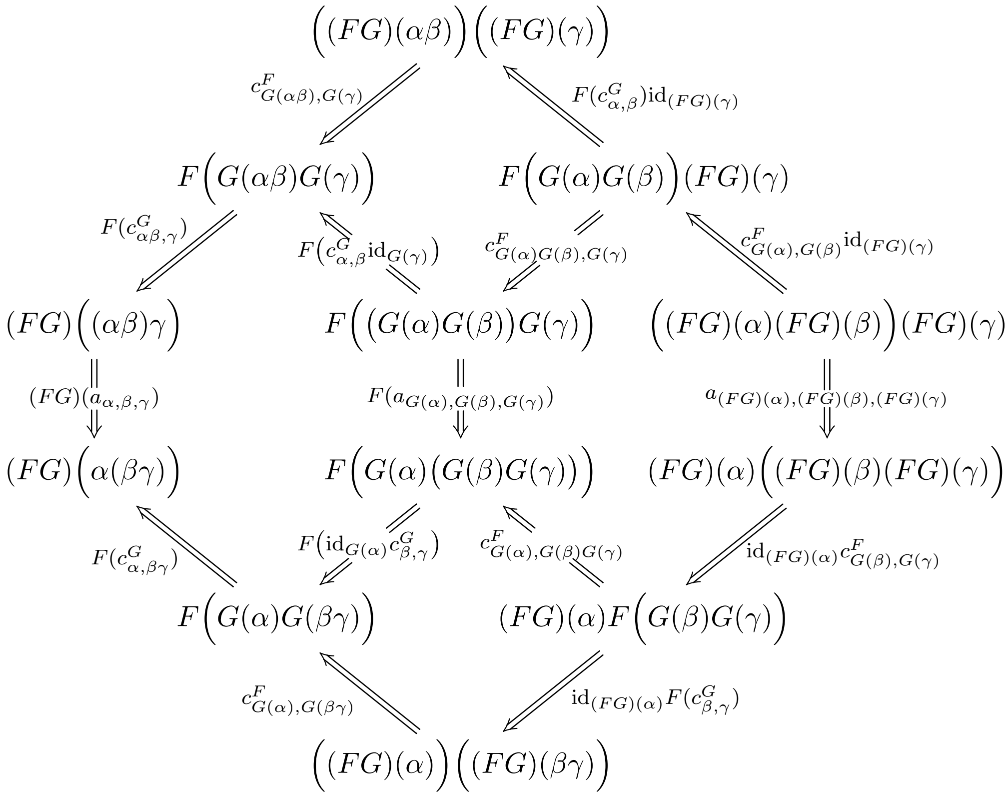
iii) For every object x in $\mathcal{C}$, the unitor $u^{F\circ G}_x$ is defined to be the vertical composite of the 2-morphisms

$$\begin{array}{c} (F\circ G)(\mathrm{id}_x) \\ \Downarrow{\scriptstyle F(u^G_x)} \\ F(\mathrm{id}_{G(x)}) \\ \Downarrow{\scriptstyle u^F_{G(x)}} \\ \mathrm{id}_{(F\circ G)(x)} \end{array} \quad . \tag{41}$$

It is not immediately clear from this definition that the data defines a 2-functor $F\circ G:\mathcal{C}\to\mathcal{E}$. A proof is therefore included to check the necessary axioms.

Proof. The properties are checked one at a time.

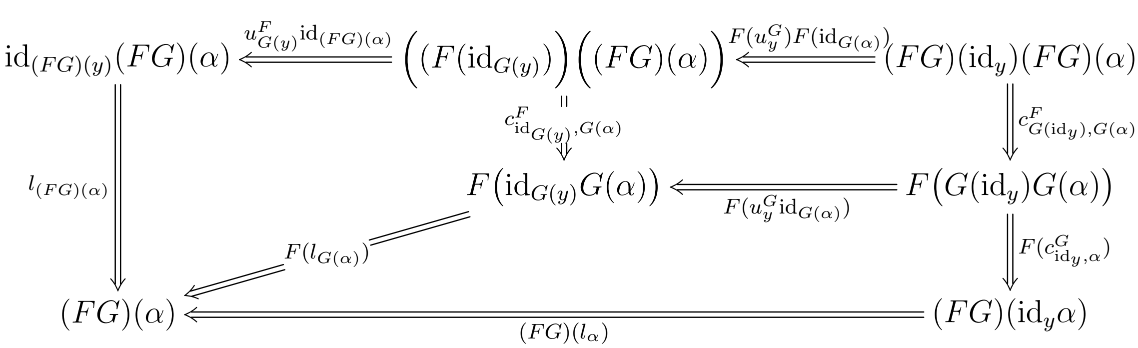
(a) Let (α, β, γ) be a triple of composable 1-morphisms in $\mathcal{C}$. The outer part of the following diagram must commute.[3]



(42)

The right hexagon commutes by condition (a) for the 2-functor F applied to the three 1-morphisms $G(\alpha), G(\beta)$, and $G(\gamma)$. The left hexagon commutes by condition (c) for the 2-functor F, associativity of vertical composition, and by condition (a) for the 2-functor G applied to the three 1-morphisms α, β, and γ. The top square commutes because $\mathrm{id}_{(FG)(\gamma)} = F(\mathrm{id}_{G(\gamma)})$ by condition (d) for the 2-functor F and by condition (e) applied to the pair $(c^G_{\alpha,\beta}, \mathrm{id}_{G(\gamma)})$. The bottom square commutes by condition (d) again and condition (e) applied to the pair $(\mathrm{id}_{G(\alpha)}, c^G_{\beta,\gamma})$. Therefore, the outer part of the diagram commutes.

(b) Let $y \xleftarrow{\alpha} x$ be a 1-morphism in $\mathcal{C}$. The outer part of the following diagram must commute.



(43)

The top right corner commutes by condition (e) for the 2-functor F applied to the pair $(u^G_y, \mathrm{id}_{G(\alpha)})$ of horizontally composable 2-morphisms. The left corner commutes by condition

[3] We've temporarily removed the composition notation $\circ$ and will continue to do so when we feel it is convenient.

(b) for the 2-functor F applied to the 1-morphism $G(y) \xleftarrow{G(\alpha)} G(x)$. The bottom corner commutes by condition (c) for the 2-functor F, associativity of vertical composition, and by condition (b) for the 2-functor G applied to the 1-morphism $y \xleftarrow{\alpha} x$. Therefore, the outer part of the diagram commutes.

A similar argument shows that the other required diagram also commutes.

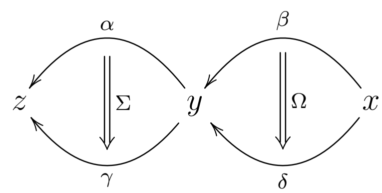
(c) Let (Σ, Ω) be a pair of vertically composable 2-morphisms in $\mathcal{C}$. Then

$$(FG)\begin{pmatrix}\Sigma\\ \Omega\end{pmatrix} = F\left(G\begin{pmatrix}\Sigma\\ \Omega\end{pmatrix}\right) = F\begin{pmatrix}G(\Sigma)\\ G(\Omega)\end{pmatrix} = \frac{F(G(\Sigma))}{F(G(\Omega))} = \frac{(FG)(\Sigma)}{(FG)(\Omega)}. \tag{44}$$

(d) Let α be a 1-morphism in $\mathcal{C}$. Then

$$(FG)(\mathrm{id}_\alpha) = F\big(G(\mathrm{id}_\alpha)\big) = F\big(\mathrm{id}_{G(\alpha)}\big) = \mathrm{id}_{F(G(\alpha))} = \mathrm{id}_{(FG)(\alpha)}. \tag{45}$$

(e) Let



(46)

be a pair of horizontally composable 2-morphisms. The outer part of the following diagram must commute.

$$\begin{array}{ccc}
F(\gamma)F(\delta) & \xLeftarrow{\big((FG)(\Sigma)\big)\big((FG)(\Omega)\big)} & \Big((FG)(\alpha)\Big)\Big((FG)(\beta)\Big) \\
{\scriptstyle c^F_{G(\gamma),G(\delta)}}\Downarrow & & \Downarrow{\scriptstyle c^F_{G(\alpha),G(\beta)}} \\
F\Big(G(\gamma)G(\delta)\Big) & \xLeftarrow[F\big(G(\Sigma)G(\Omega)\big)]{} & F\Big(G(\alpha)G(\beta)\Big) \\
{\scriptstyle F(c^G_{\gamma,\delta})}\Downarrow & & \Downarrow{\scriptstyle F(c^G_{\alpha,\beta})} \\
(FG)(\gamma\delta) & \xLeftarrow[(FG)(\Sigma\Omega)]{} & (FG)(\alpha\beta)
\end{array} \tag{47}$$

The top square commutes by condition (e) for the 2-functor F applied to the pair $(G(\Sigma), G(\Omega))$. The bottom square commutes by condition (c) for the 2-functor F and by condition (e) for the 2-functor G applied to the pair (Σ, Ω). Therefore, the outer part of the diagram commutes.

■

At this point, a natural question to ask is whether the composition of 2-functors is associative. It is also not immediately obvious whether or not the composition with the identity 2-functor doesn't change the 2-functor that it is composed with. However, in order to properly discuss this, pseudonatural transformations and pseudonatural equivalences must be introduced.

Definition 48. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be two 2-functors. A <u>*psuedonatural transformation*</u> ρ from F to G, written as $\rho : F \Rightarrow G$, consists of the following data:

i) a function $\rho : C_0 \to D_1$ assigning a 1-morphism to an object x in the following manner

$$x \quad \stackrel{\rho}{\longmapsto} \quad \begin{array}{c} F(x) \\ \big\downarrow {\scriptstyle \rho(x)} \\ G(x) \end{array} \tag{49}$$

ii) and a function $\rho : C_1 \to D_2$ assigning a vertically invertible 2-morphism to every 1-morphism $y \xleftarrow{\alpha} x$ in the following manner

$$y \xleftarrow{\ \alpha\ } x \quad \stackrel{\rho}{\longmapsto} \quad \begin{array}{ccc} F(y) & \xleftarrow{F(\alpha)} & F(x) \\ {\scriptstyle \rho(y)}\big\downarrow & \overset{\rho(\alpha)}{\Searrow} & \big\downarrow {\scriptstyle \rho(x)} \\ G(y) & \xleftarrow[G(\alpha)]{} & G(x) \end{array} . \tag{50}$$

These data must satisfy the following conditions.

(a) For every pair $(z \xleftarrow{\alpha} y, y \xleftarrow{\beta} x)$ of composable 1-morphisms in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} \Big(\rho(z)F(\alpha)\Big)F(\beta) & \xRightarrow{\rho(\alpha)\mathrm{id}_{F(\beta)}} & \Big(G(\alpha)\rho(y)\Big)F(\beta) \\ {\scriptstyle a_{\rho(z),F(\alpha),F(\beta)}}\Big\Downarrow & & \Big\Downarrow {\scriptstyle a_{G(\alpha),\rho(y),F(\beta)}} \\ \rho(z)\Big(F(\alpha)F(\beta)\Big) & & G(\alpha)\Big(\rho(y)F(\beta)\Big) \\ {\scriptstyle \mathrm{id}_{\rho(z)}c^F_{\alpha,\beta}}\Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{id}_{G(\alpha)}\rho(\beta)} \\ \rho(z)F(\alpha\beta) & & G(\alpha)\Big(G(\beta)\rho(x)\Big) \\ {\scriptstyle \rho(\alpha\beta)}\Big\Downarrow & & \Big\Downarrow {\scriptstyle a^{-1v}_{G(\alpha),G(\beta),\rho(x)}} \\ G(\alpha\beta)\rho(x) & \xLeftarrow{c^G_{\alpha,\beta}\mathrm{id}_{\rho(x)}} & \Big(G(\alpha)G(\beta)\Big)\rho(x) \end{array} \tag{51}$$

commutes.

(b) For every 2-morphism

$$y \underset{\gamma}{\overset{\alpha}{\Longleftarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\;\Downarrow\Sigma}} x\ , \tag{52}$$

the diagram

$$\begin{array}{ccc} G(\alpha)\rho(x) & \xLeftarrow{\rho(\alpha)} & \rho(y)F(\alpha) \\ {\scriptstyle G(\Sigma)\mathrm{id}_{\rho(x)}}\Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{id}_{\rho(y)}F(\Sigma)} \\ G(\gamma)\rho(x) & \xLeftarrow[\rho(\gamma)]{} & \rho(y)F(\gamma) \end{array} \tag{53}$$

commutes.

Remark 54. There was no condition on ρ in the previous definition for the identity 1-morphism $x \xleftarrow{\mathrm{id}_x} x$ for an object x of $\mathcal{C}$. This condition would require that the diagram

$$
\begin{array}{ccccc}
\rho(x)\mathrm{id}_{F(x)} & & \xLeftarrow{\mathrm{id}_{\rho(x)} u^F_x} & & \rho(x)F(\mathrm{id}_x) \\
{\scriptstyle r_{\rho(x)}}\Downarrow & & & & \Downarrow{\scriptstyle \rho(\mathrm{id}_x)} \\
\rho(x) & \xLeftarrow[l_{\rho(x)}]{} & \mathrm{id}_{G(x)}\rho(x) & \xLeftarrow[u^G_x \mathrm{id}_{\rho(x)}]{} & G(\mathrm{id}_x)\rho(x)
\end{array}
\tag{55}
$$

commute. We leave this verification to the enthusiast.

Definition 56. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and let $F, G, H : \mathcal{C} \to \mathcal{D}$ be three 2-functors and let $\rho : F \Rightarrow G$ and $\sigma : G \Rightarrow H$ be two pseudonatural transformations. The *vertical composition* of ρ with σ, written as $\substack{\rho \\ \sigma} : F \Rightarrow H$ is defined as follows.

i) To every object x in $\mathcal{C}$, assign

$$
x \quad \xmapsto{\substack{\rho \\ \sigma}} \quad
\begin{array}{c}
F(x) \\
\downarrow{\scriptstyle \rho(x)} \\
G(x) \\
\downarrow{\scriptstyle \sigma(x)} \\
H(x)
\end{array}
\ ,
\tag{57}
$$

the composition $\sigma(x)\rho(x)$.

ii) To every 1-morphism $y \xleftarrow{\alpha} x$ in $\mathcal{C}$, assign the 2-morphism

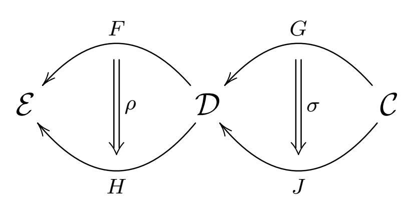
$$
\Big(\sigma(y)\rho(y)\Big)F(\alpha) \xRightarrow{\substack{\rho \\ \sigma}(\alpha)} H(\alpha)\Big(\sigma(x)\rho(x)\Big)
\tag{58}
$$

defined by the vertical composition of the following 2-morphisms in $\mathcal{D}$

$$
\begin{array}{ccccc}
\Big(\sigma(y)\rho(y)\Big)F(\alpha) & \xRightarrow{a_{\sigma(y),\rho(y),F(\alpha)}} & \sigma(y)\Big(\rho(y)F(\alpha)\Big) & \xRightarrow{\mathrm{id}_{\sigma(y)}\rho(\alpha)} & \sigma(y)\Big(G(\alpha)\rho(x)\Big) \\
& & & & \Downarrow{\scriptstyle a^{-1_v}_{\sigma(y),G(\alpha),\rho(x)}} \\
H(\alpha)\Big(\sigma(x)\rho(x)\Big) & \xLeftarrow[a_{H(\alpha),\sigma(x),\rho(x)}]{} & \Big(H(\alpha)\sigma(x)\Big)\rho(x) & \xLeftarrow[\sigma(\alpha)\mathrm{id}_{\rho(x)}]{} & \Big(\sigma(y)G(\alpha)\Big)\rho(x)
\end{array}
\ .
\tag{59}
$$

Again, it is not obvious that this definition of vertical composition of pseudonatural transformations results in another pseudonatural transformation. We leave it to the reader to check that conditions (a) and (b) hold. Instead, we move on to discussing the horizontal composition of pseudonatural transformations.

Definition 60. Consider a collection of 2-categories, 2-functors, and pseudonatural transformations fitting into a diagram of the form



(61)

The *horizontal composition* of ρ with σ, written as $\rho\sigma : FG \Rightarrow HJ$ is defined as follows.

i) To every object x in $\mathcal{C}$, assign the composition

$$x \xmapsto{\rho\sigma} \begin{array}{ccc} F(J(x)) & \xleftarrow{F(\sigma(x))} & F(G(x)) \\ \big\downarrow{\scriptstyle \rho(J(x))} & & \\ H(J(x)) & & \end{array} \tag{62}$$

ii) To every 1-morphism $y \xleftarrow{\alpha} x$ in $\mathcal{C}$, assign the 2-morphism

$$\Big((\rho\sigma)(y)\Big)\Big((FG)(\alpha)\Big) \xRightarrow{(\rho\sigma)(\alpha)} \Big((HJ)(\alpha)\Big)\Big((\rho\sigma)(x)\Big) \tag{63}$$

defined by the vertical composition of the following 2-morphisms in $\mathcal{D}$

$$\begin{array}{ccc} \rho(J(y))\Big(F(\sigma(y))(FG)(\alpha)\Big) & \xLeftarrow{a_{\rho(J(y)),F(\sigma(y)),(FG)(\alpha)}} & \Big(\rho(J(y))F(\sigma(y))\Big)(FG)(\alpha) \\ \Big\Downarrow{\scriptstyle \mathrm{id}_{\rho(J(y))}c^F_{\sigma(y),G(\alpha)}} & & \\ \rho(J(y))F(\sigma(y)G(\alpha)) & \xRightarrow{\mathrm{id}_{\rho(J(y))}F(\sigma(\alpha))} & \rho(J(y))F(J(\alpha)\sigma(x)) \\ & & \Big\Downarrow{\scriptstyle \mathrm{id}_{\rho(J(y))}(c^F_{J(\alpha),\sigma(x)})^{-1v}} \\ \Big(\rho(J(y))(FJ)(\alpha)\Big)F(\sigma(x)) & \xLeftarrow[a^{-1v}_{\rho(J(y)),(FJ)(\alpha),F(\sigma(x))}]{} & \rho(J(y))\Big((FJ)(\alpha)F(\sigma(x))\Big) \\ \Big\Downarrow{\scriptstyle \rho(J(\alpha))\mathrm{id}_{F(\sigma(x))}} & & \\ \Big((HJ)(\alpha)\rho(J(x))\Big)F(\sigma(x)) & \xRightarrow[a_{(HJ)(\alpha),\rho(J(x)),F(\sigma(x))}]{} & (HJ)(\alpha)\Big(\rho(J(x))F(\sigma(x))\Big) \end{array} . \tag{64}$$

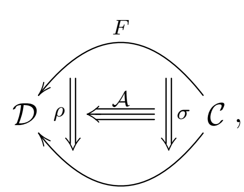
Remark 65. There are actually two natural choices for the composition of pseudonatural transformations. The other one involves assigning to every object x of $\mathcal{C}$

$$x \xmapsto{\rho\sigma} \begin{array}{ccc} & & F(G(x)) \\ & & \big\downarrow{\scriptstyle \rho(G(x))} \\ H(J(x)) & \xleftarrow{H(\sigma(x))} & H(G(x)) \end{array} \tag{66}$$

and a similar adjustment for the assignment on morphisms. By the existence of the vertical isomorphism $\sigma(\rho(x)) : \rho(J(x))F(\sigma(x)) \Rightarrow H(\sigma(x))\rho(G(x))$, these two results are isomorphic. We will not discuss in more detail the relationship between the two and we will stick with the first definition.

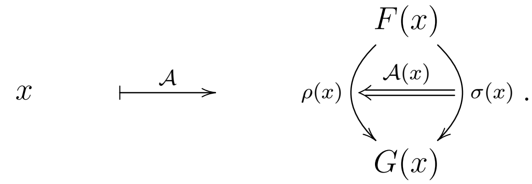
As before, one should check that the definition above indeed defines a pseudonatural transformation. Again, we skip the proof.

Definition 67. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories, $F, G : \mathcal{C} \to \mathcal{D}$ be two 2-functors, and $\rho, \sigma : F \Rightarrow G$ be two pseudonatural transformations. A *modification* $\mathcal{A}$ from σ to ρ, written as $\mathcal{A} : \sigma \Rrightarrow \rho$ and drawn as



(68)

consists of a function $\mathcal{A} : C_0 \to D_2$ assigning a 2-morphism to an object x in the following manner



(69)

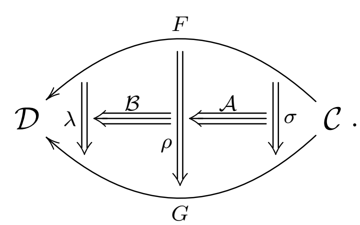
This assignment must satisfy the condition that for every 1-morphism $y \xleftarrow{\alpha} x$, the diagram

$$\begin{array}{ccc} G(\alpha)\sigma(x) & \xLeftarrow{\sigma(\alpha)} & \sigma(y)F(\alpha) \\ {\scriptstyle \mathrm{id}_{G(\alpha)}\mathcal{A}(x)} \Downarrow & & \Downarrow {\scriptstyle \mathcal{A}(y)\mathrm{id}_{F(\alpha)}} \\ G(\alpha)\rho(x) & \xLeftarrow[\rho(\alpha)]{} & \rho(y)F(\alpha) \end{array} \tag{70}$$

commutes.

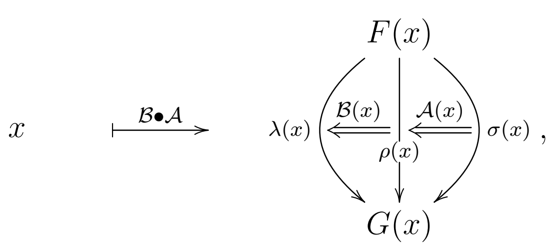
Modifications have three types of compositions.

Definition 71. Consider 2-categories, 2-functors, pseudonatural transformations, and modifications as in the following diagram



(72)

The <u>*internal composition*</u> of $\mathcal{A}$ and $\mathcal{B}$, written as $\mathcal{B} \bullet \mathcal{A} : \sigma \Rrightarrow \lambda$, is the modification defined by the assignment



i.e. $(\mathcal{B} \bullet \mathcal{A})(x) := \begin{matrix} \mathcal{A}(x) \\ \mathcal{B}(x) \end{matrix}$. (73)

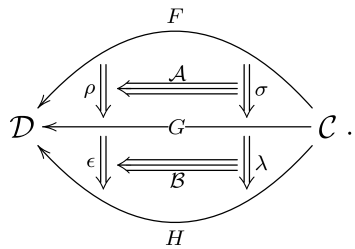
The internal composition of two modifications is indeed a modification.

Proof. Let $y \xleftarrow{\alpha} x$ be a 1-morphism in $\mathcal{C}$. The outer part of the following diagram should commute.

$$\begin{array}{ccc} G(\alpha)\sigma(x) & \xLeftarrow{\sigma(\alpha)} & \sigma(y)F(\alpha) \\ {\scriptstyle \mathrm{id}_{G(\alpha)}\mathcal{A}(x)} \Downarrow & & \Downarrow {\scriptstyle \mathcal{A}(y)\mathrm{id}_{F(\alpha)}} \\ G(\alpha)\rho(x) & \xLeftarrow[\rho(\alpha)]{} & \rho(y)F(\alpha) \\ {\scriptstyle \mathrm{id}_{G(\alpha)}\mathcal{B}(x)} \Downarrow & & \Downarrow {\scriptstyle \mathcal{B}(y)\mathrm{id}_{F(\alpha)}} \\ G(\alpha)\lambda(x) & \xLeftarrow[\lambda(\alpha)]{} & \lambda(y)F(\alpha) \end{array} \tag{74}$$

Each square commutes by the only condition for $\mathcal{A}$ and $\mathcal{B}$ being modifications and by the interchange law in $\mathcal{D}$. ■

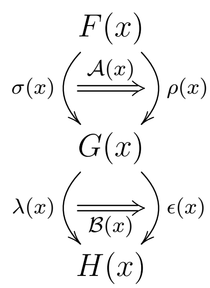
Definition 75. Consider 2-categories, 2-functors, pseudonatural transformations, and modifications as in the following diagram



(76)

The <u>*vertical composition*</u> of $\mathcal{A}$ and $\mathcal{B}$, written as $\begin{matrix}\mathcal{A}\\ \circ\\ \mathcal{B}\end{matrix} : \begin{matrix}\sigma\\ \circ\\ \lambda\end{matrix} \Rrightarrow \begin{matrix}\rho\\ \circ\\ \epsilon\end{matrix}$, is the modification defined by the assignment

$x \xmapsto{\begin{matrix}\mathcal{A}\\ \circ\\ \mathcal{B}\end{matrix}}$

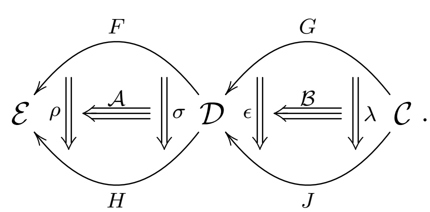


, i.e. $\begin{matrix}\mathcal{A}\\ \circ\\ \mathcal{B}\end{matrix}(x) := \mathcal{B}(x)\mathcal{A}(x),$ (77)

the horizontal composition of 2-morphisms in $\mathcal{D}$.

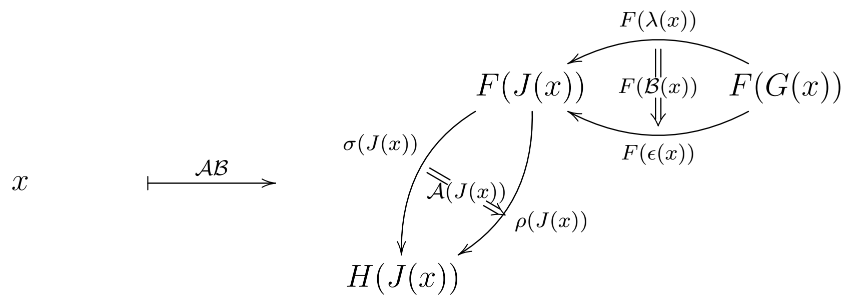
Remark 78. This notation is unfortunately confusing but is essentially the same abusive notation as in Definition 56. The modification was defined using vertical compositions of natural transformations but the actual definition involved horizontal composition in the 2-category $\mathcal{D}$. The reader is encouraged to draw more pictures of diagrams to avoid further confusion.

Definition 79. Consider 2-categories, 2-functors, pseudonatural transformations, and modifications as in the following diagram



(80)

The <u>*horizontal composition*</u> of $\mathcal{A}$ and $\mathcal{B}$, written as $\mathcal{A} \circ \mathcal{B} : \sigma \circ \lambda \Rrightarrow \rho \circ \epsilon$, is the modification defined by the assignment



(81)

i.e.

$$(\mathcal{A} \circ \mathcal{B})(x) := \Big(\mathcal{A}(J(x))\Big)\Big(F(\mathcal{B}(x))\Big) \tag{82}$$

for all objects x in $\mathcal{C}$.

We now come back to answering the question posed about the associativity, or lack thereof, of composition of 2-functors and pseudonatural transformations.

Lemma 83. *Consider the following sequence of 2-categories and 2-functors*

$$\mathcal{F} \xleftarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D} \xleftarrow{H} \mathcal{C}. \tag{84}$$

Then $(FG)H = F(GH)$, i.e. the composition of 2-functors is associative, or equivalently the associator (a-priori a nontrivial pseudonatural transformation) for 2-functor composition is the identity.

Proof. We prove this by checking all the data that specify the 2-functors $(FG)H$ and $F(GH)$ are equal (see Definition 38).

i) Because ordinary composition of functions is associative,

$$((FG)H)_i = (FG)_i H_i = (F_i G_i) H_i = F_i(G_i H_i) = F_i(GH)_i = (F(GH))_i. \tag{85}$$

ii) For every pair (α, β) of 1-morphisms in $\mathcal{C}$, we have the following list of equalities

$$c^{(FG)H}_{\alpha,\beta} = \begin{matrix} c^{FG}_{H(\alpha),H(\beta)} \\ (FG)(c^{H}_{\alpha,\beta}) \end{matrix} = \begin{matrix} \left(\begin{matrix} c^{F}_{G(H(\alpha)),G(H(\beta))} \\ F(c^{G}_{H(\alpha),H(\beta)}) \end{matrix}\right) \\ (FG)(c^{H}_{\alpha,\beta}) \end{matrix} = \begin{matrix} c^{F}_{G(H(\alpha)),G(H(\beta))} \\ \left(\begin{matrix} F(c^{G}_{H(\alpha),H(\beta)}) \\ (FG)(c^{H}_{\alpha,\beta}) \end{matrix}\right) \end{matrix} = \begin{matrix} c^{F}_{G(H(\alpha)),G(H(\beta))} \\ F\left(\begin{matrix} c^{G}_{H(\alpha),H(\beta)} \\ G(c^{H}_{\alpha,\beta}) \end{matrix}\right) \end{matrix} = \begin{matrix} c^{F}_{(GH)(\alpha),(GH)(\beta)} \\ F(c^{GH}_{\alpha,\beta}) \end{matrix} = c^{F(GH)}_{\alpha,\beta} \tag{86}$$

because vertical composition is associative and because 2-functors respect vertical composition.

iii) For every object x in $\mathcal{C}$, we have the following list of equalities

$$u_x^{(FG)H} = \begin{matrix} (FG)(u_x^H) \\ u^{FG}_{H(x)} \end{matrix} = \begin{matrix} (FG)(u_x^H) \\ \begin{pmatrix} F(u^G_{H(x)}) \\ u^F_{G(H(x))} \end{pmatrix} \end{matrix} = \begin{matrix} \begin{pmatrix} (FG)(u_x^H) \\ F(u^G_{H(x)}) \end{pmatrix} \\ u^F_{G(H(x))} \end{matrix} = \begin{matrix} F\begin{pmatrix} G(u_x^H) \\ u^G_{H(x)} \end{pmatrix} \\ u^F_{G(H(x))} \end{matrix} = \begin{matrix} F(u_x^{GH}) \\ u^F_{(GH)(x)} \end{matrix} = u_x^{F(GH)} \tag{87}$$

because vertical composition is associative and because 2-functors respect vertical composition.

■

Definition 88. Let $\mathcal{C}$ be a 2-category. The *identity 2-functor* on $\mathcal{C}$, written as $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, is defined as follows.

i) The assignment on objects, 1-morphisms, and 2-morphisms is given by the identity functions

$$(\mathrm{id}_{\mathcal{C}})_j := \Big(\mathrm{id}_{C_j} : C_j \to C_j\Big) \tag{89}$$

for all $j = 0, 1, 2$.

ii) For every pair (α, β) of composable 1-morphisms in $\mathcal{C}$, the compositor is

$$c^{\mathrm{id}_{\mathcal{C}}}_{\alpha,\beta} := \mathrm{id}_{\alpha\beta}, \tag{90}$$

the identity 2-morphism on $\alpha\beta$.

iii) For every object x in $\mathcal{C}$, the unitor is

$$u_x^{\mathrm{id}_{\mathcal{C}}} := \mathrm{id}_{\mathrm{id}_x} \tag{91}$$

the identity 2-morphism on id_x.

Lemma 92. *Let $F : \mathcal{C} \to \mathcal{D}$ be a 2-functor between two 2-categories $\mathcal{C}$ and $\mathcal{D}$. Then $F\mathrm{id}_{\mathcal{C}} = F = \mathrm{id}_{\mathcal{D}}F$, i.e. the left and right unifiers for the composition of 2-functors are both* equal *to the identity.*

Proof. This is proved similarly to the previous Lemma.

i) Because ordinary composition of functions by an identity function results in that same function,

$$(F\mathrm{id}_{\mathcal{C}})_j = F_j(\mathrm{id}_{\mathcal{C}})_j = F_j = (\mathrm{id}_{\mathcal{D}})_j F_j = (\mathrm{id}_{\mathcal{D}}F)_j \tag{93}$$

for all $j = 0, 1, 2$.

ii) For every pair (α, β) of 1-morphisms in $\mathcal{C}$, we have the following list of equalities

$$c^{F\mathrm{id}_\mathcal{C}}_{\alpha,\beta} = \begin{matrix} c^F_{\mathrm{id}_\mathcal{C}(\alpha),\mathrm{id}_\mathcal{C}(\beta)} \\ F(c^{\mathrm{id}_\mathcal{C}}_{\alpha,\beta}) \end{matrix} = \begin{matrix} c^F_{\alpha,\beta} \\ F(\mathrm{id}_{\alpha\beta}) \end{matrix} = \begin{matrix} c^F_{\alpha,\beta} \\ \mathrm{id}_{F(\alpha\beta)} \end{matrix} = c^F_{\alpha,\beta} = \begin{matrix} \mathrm{id}_{F(\alpha)F(\beta)} \\ c^F_{\alpha,\beta} \end{matrix} = \begin{matrix} c^{\mathrm{id}_\mathcal{D}}_{F(\alpha),F(\beta)} \\ \mathrm{id}_\mathcal{D}(c^F_{\alpha,\beta}) \end{matrix} = c^{\mathrm{id}_\mathcal{D}F}_{\alpha,\beta} \tag{94}$$

because the vertical identity 2-morphism is an identity for vertical composition and because 2-functors respect vertical identities.

iii) For every object x in $\mathcal{C}$, we have the following list of equalities

$$u^{F\mathrm{id}_\mathcal{C}}_x = \begin{matrix} F(u^{\mathrm{id}_\mathcal{C}}_x) \\ u^F_{\mathrm{id}_\mathcal{C}(x)} \end{matrix} = \begin{matrix} F(\mathrm{id}_{\mathrm{id}_x}) \\ u^F_x \end{matrix} = u^F_x = \begin{matrix} u^F_x \\ \mathrm{id}_{\mathrm{id}_{F(x)}} \end{matrix} = \begin{matrix} \mathrm{id}_\mathcal{D}(u^F_x) \\ u^{\mathrm{id}_\mathcal{D}}_{F(x)} \end{matrix} = u^{\mathrm{id}_\mathcal{D}F}_x \tag{95}$$

because the vertical identity 2-morphism is an identity for vertical composition and because 2-functors respect vertical identities.

■

Although composition of 2-functors has no surprises, vertical composition of pseudonatural transformations is a bit more complicated. In particular, there are associators and unifiers.

Lemma 96. *Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories, $F, G, H, J : \mathcal{C} \to \mathcal{D}$ be four 2-functors, and $\rho : F \Rightarrow G$, $\sigma : G \Rightarrow H$, and $\lambda : H \Rightarrow J$ be three pseudonatural transformations. Then the assignment*

$$x \mapsto a_{\lambda,\sigma,\rho}(x) := a_{\lambda(x),\sigma(x),\rho(x)}, \tag{97}$$

the associator in the category $\mathcal{D}$, for any object x in $\mathcal{C}$, defines a modification

$$a_{\lambda,\sigma,\rho} : \begin{matrix} \rho \\ \begin{pmatrix} \sigma \\ \lambda \end{pmatrix} \end{matrix} \Rrightarrow \begin{matrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \\ \lambda \end{matrix}. \tag{98}$$

Furthermore, $a_{\rho,\sigma,\lambda}$ is invertible and satisfies the pentagon axiom of condition (c) in Definition 2.

Definition 99. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a 2-functor. The *identity pseudonatural transformation* $\mathrm{id}_F : F \Rightarrow F$ is defined as follows.

i) The assignment

$$x \mapsto \Big(\mathrm{id}_{F(x)} : F(x) \to F(x)\Big) \tag{100}$$

defines the map $\mathrm{id}_F : C_0 \to D_1$.

ii) The assignment

$$\left(y \xleftarrow{\alpha} x\right) \mapsto \left(\substack{r_{F(\alpha)} \\ l^{-1v}_{F(\alpha)}} : \mathrm{id}_{F(y)}F(\alpha) \Rightarrow F(a) \Rightarrow F(\alpha)\mathrm{id}_{F(x)} \right) \tag{101}$$

defines $\mathrm{id}_F : C_1 \to D_2$.

Lemma 102. *Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories, $F, G : \mathcal{C} \to \mathcal{D}$ be two 2-functors, and $\rho : F \Rightarrow G$ a pseudonatural transformation. Then the assignments*

$$x \mapsto l_\rho(x) := l_{\rho(x)}, \tag{103}$$

the left unifier in $\mathcal{D}$, and

$$x \mapsto r_\rho(x) := r_{\rho(x)}, \tag{104}$$

the right unifier in $\mathcal{D}$, define modifications

$$l_\rho : \begin{smallmatrix}\mathrm{id}_F \\ \rho\end{smallmatrix} \Rrightarrow \rho \tag{105}$$

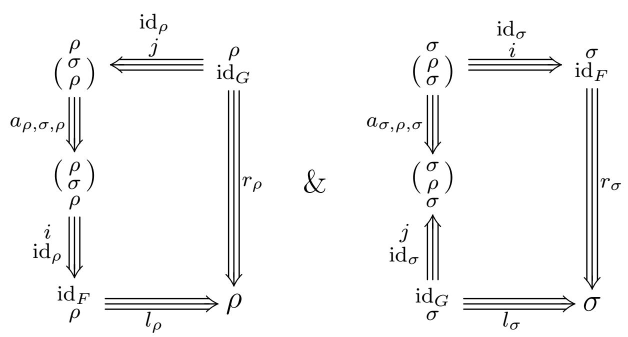
and

$$r_\rho : \begin{smallmatrix}\rho \\ \mathrm{id}_F\end{smallmatrix} \Rrightarrow \rho \tag{106}$$

respectively. Furthermore, l_ρ and r_ρ are invertible and satisfy condition (d) in Definition 2.

Finally, after giving the numerous definitions that arise in basic 2-category theory, we come to the notion of a psuedonatural equivalence of 2-functors between two 2-categories.

Definition 107. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and $F, G : \mathcal{C} \to \mathcal{D}$ two 2-functors. A *pseudonatural equivalence* from F to G is a quadruple (ρ, σ, i, j) consisting of pseudonatural transformations $\rho : F \Rightarrow G$, $\sigma : G \Rightarrow F$, and invertible modifications $i : \begin{smallmatrix}\rho \\ \sigma\end{smallmatrix} \Rrightarrow \mathrm{id}_F$ and $j : \mathrm{id}_G \Rrightarrow \begin{smallmatrix}\sigma \\ \rho\end{smallmatrix}$ such that the diagrams



(108)

both commute. In these diagrams, l and r are the left and right unifiers, respectively, of Lemma 102 and a is the associator of Lemma 96. The above two diagrams are known as *zig-zag identities* (for reasons which we will not go into here). We abusively say that $\rho : F \Rightarrow G$ is a pseudonatural equivalence without writing out the other pieces of data. F and G are said to be *pseudonaturally equivalent* if there exists a pseudonatural equivalence between them.

Definition 109. Let S and T be two 2-categories and $F : S \to T$ a 2-functor. F is called an *equivalence of 2-categories* if there exists a functor $G : T \to S$ together with pseudonatural equivalences $\rho_S : GF \Rightarrow \mathrm{id}_S$ and $\rho_T : FG \Rightarrow \mathrm{id}_T$. The functor $G : T \to S$ along with the pseudonatural equivalences is called a *weak inverse* of F.

Lemma 110. *Let S and T be two 2-categories. Two weak inverses to a 2-functor $F : S \to T$ are pseudonaturally equivalent.*

Proof. By assumption, there exist $G, G' : T \to S$ with pseudonatural equivalences

$$\left(\rho_S : GF \Rightarrow \mathrm{id}_S,\ \sigma_S : \mathrm{id}_S \Rightarrow GF,\ i_S : \begin{smallmatrix}\rho_S\\ \sigma_S\end{smallmatrix} \Rrightarrow \mathrm{id}_{GF},\ j_S : \mathrm{id}_{\mathrm{id}_S} \Rrightarrow \begin{smallmatrix}\sigma_S\\ \rho_S\end{smallmatrix}\right), \tag{111}$$

$$\left(\rho_T : FG \Rightarrow \mathrm{id}_T,\ \sigma_T : \mathrm{id}_T \Rightarrow FG,\ i_T : \begin{smallmatrix}\rho_T\\ \sigma_T\end{smallmatrix} \Rrightarrow \mathrm{id}_{FG},\ j_T : \mathrm{id}_{\mathrm{id}_T} \Rrightarrow \begin{smallmatrix}\sigma_T\\ \rho_T\end{smallmatrix}\right), \tag{112}$$

$$\left(\rho'_S : G'F \Rightarrow \mathrm{id}_S,\ \sigma'_S : \mathrm{id}_S \Rightarrow G'F,\ i'_S : \begin{smallmatrix}\rho'_S\\ \sigma'_S\end{smallmatrix} \Rrightarrow \mathrm{id}_{G'F},\ j'_S : \mathrm{id}_{\mathrm{id}_S} \Rrightarrow \begin{smallmatrix}\sigma'_S\\ \rho'_S\end{smallmatrix}\right), \tag{113}$$

and

$$\left(\rho'_T : FG' \Rightarrow \mathrm{id}_T,\ \sigma'_T : \mathrm{id}_T \Rightarrow FG',\ i'_T : \begin{smallmatrix}\rho'_T\\ \sigma'_T\end{smallmatrix} \Rrightarrow \mathrm{id}_{FG'},\ j'_T : \mathrm{id}_{\mathrm{id}_T} \Rrightarrow \begin{smallmatrix}\sigma'_T\\ \rho'_T\end{smallmatrix}\right). \tag{114}$$

We define a pseudonatural transformation $\rho : G \Rightarrow G'$ by taking the vertical composition of pseudonatural transformations

$$G = \mathrm{id}_S G \xRightarrow{\sigma'_S \mathrm{id}_G} (G'F)G = G'(FG) \xRightarrow{\mathrm{id}_{G'}\rho_T} G'\mathrm{id}_T = G' \tag{115}$$

and a psuedonatural transformation $\sigma : G' \Rightarrow G$ by

$$G' = G'\mathrm{id}_T \xRightarrow{\mathrm{id}_{G'}\sigma_T} G'(FG) = (G'F)G \xRightarrow{\rho'_S \mathrm{id}_G} \mathrm{id}_S G = G, \tag{116}$$

both of which have been simplified by Lemma 83 and Lemma 92. We define a modification $i : \begin{smallmatrix}\rho\\ \sigma\end{smallmatrix} \Rrightarrow \mathrm{id}_G$ by the internal composition

$$\begin{array}{ccc}
\begin{smallmatrix}\rho\\ \sigma\end{smallmatrix} \equiv \begin{matrix}\begin{pmatrix}\sigma'_S\mathrm{id}_G\\ \mathrm{id}_{G'}\rho_T\end{pmatrix}\\ \begin{pmatrix}\mathrm{id}_{G'}\sigma_T\\ \rho'_S\mathrm{id}_G\end{pmatrix}\end{matrix} & \overset{\begin{pmatrix}a_{\mathrm{id}_{G'}\sigma_T,\mathrm{id}_{G'}\rho_T,\sigma'_S\mathrm{id}_G}\\ \mathrm{id}_{\rho'_S\mathrm{id}_G}\end{pmatrix}\bullet a_{\rho'_S\mathrm{id}_G,\mathrm{id}_{G'}\sigma_T,\begin{smallmatrix}\sigma'_S\mathrm{id}_G\\ \mathrm{id}_{G'}\rho_T\end{smallmatrix}}}{\Rrightarrow} & \begin{pmatrix}\sigma'_S\mathrm{id}_G\\ \begin{pmatrix}\mathrm{id}_{G'}\rho_T\\ \mathrm{id}_{G'}\sigma_T\end{pmatrix}\\ \rho'_S\mathrm{id}_G\end{pmatrix} \\
 & & \Big\Downarrow \begin{pmatrix}\mathrm{id}_{\sigma'_S\mathrm{id}_G}\\ \mathrm{id}_{\mathrm{id}_{G'}}i_T\\ \mathrm{id}_{\rho'_S\mathrm{id}_G}\end{pmatrix} \\
\mathrm{id}_G \equiv \mathrm{id}_{\mathrm{id}_S}\mathrm{id}_G \underset{j'^{-1}_S\mathrm{id}_{\mathrm{id}_G}}{\Lleftarrow} \begin{matrix}\sigma'_S\mathrm{id}_G\\ \rho'_S\mathrm{id}_G\end{matrix} & \underset{\begin{smallmatrix}r_{\sigma'_S\mathrm{id}_G}\\ \mathrm{id}_{\rho'_S\mathrm{id}_G}\end{smallmatrix}}{\Lleftarrow} & \begin{pmatrix}\sigma'_S\mathrm{id}_G\\ \mathrm{id}_{G'FG}\\ \rho'_S\mathrm{id}_G\end{pmatrix} \equiv \begin{pmatrix}\sigma'_S\mathrm{id}_G\\ \mathrm{id}_{G'}\mathrm{id}_{FG}\\ \rho'_S\mathrm{id}_G\end{pmatrix}
\end{array} \tag{117}$$

and a modification $j : \mathrm{id}_{G'} \Rrightarrow \begin{smallmatrix}\sigma\\ \rho\end{smallmatrix}$ by the composition

$$
\begin{array}{ccccc}
\begin{smallmatrix}\sigma\\ \rho\end{smallmatrix} \equiv \begin{pmatrix}\mathrm{id}_{G'}\sigma_T\\ \rho'_S\mathrm{id}_G\end{pmatrix}\!\!\begin{pmatrix}\sigma'_S\mathrm{id}_G\\ \mathrm{id}_{G'}\rho_T\end{pmatrix} & \xLeftarrow{\quad a_{\begin{smallmatrix}\sigma'_S\mathrm{id}_G\\ \mathrm{id}_{G'}\rho_T\end{smallmatrix},\rho'_S\mathrm{id}_G,\mathrm{id}_{G'}\sigma_T} \bullet \begin{pmatrix}\mathrm{id}_{\mathrm{id}_{G'}\sigma_T}\\ a^{-1}_{\mathrm{id}_{G'}\rho_T,\sigma'_S\mathrm{id}_G,\rho'_S\mathrm{id}_G}\end{pmatrix}\quad} & & & \begin{pmatrix}\mathrm{id}_{G'}\sigma_T\\ \begin{pmatrix}\rho'_S\mathrm{id}_G\\ \sigma'_S\mathrm{id}_G\end{pmatrix}\\ \mathrm{id}_{G'}\rho_T\end{pmatrix} \\
& & & \begin{pmatrix}\mathrm{id}_{\mathrm{id}_{G'}\sigma_T}\\ i'^{-1}_S\mathrm{id}_{\mathrm{id}_G}\\ \mathrm{id}_{\mathrm{id}_{G'}\rho_T}\end{pmatrix} & \Uparrow \\
\mathrm{id}_G \equiv \mathrm{id}_{G'}\mathrm{id}_{\mathrm{id}_T} \xRrightarrow[\mathrm{id}_{\mathrm{id}_{G'}}j_T]{} & \begin{matrix}\mathrm{id}_{G'}\sigma_T\\ \mathrm{id}_{G'}\rho_T\end{matrix} & \xRrightarrow[\begin{smallmatrix}\mathrm{id}_{\mathrm{id}_{G'}\sigma_T}\\ l^{-1}_{\mathrm{id}_{G'}\rho_T}\end{smallmatrix}]{} & \begin{pmatrix}\mathrm{id}_{G'}\sigma_T\\ \mathrm{id}_{\mathrm{id}_S G}\\ \mathrm{id}_{G'}\rho_T\end{pmatrix} \equiv & \begin{pmatrix}\mathrm{id}_{G'}\sigma_T\\ \mathrm{id}_{\mathrm{id}_S}\mathrm{id}_G\\ \mathrm{id}_{G'}\rho_T\end{pmatrix}
\end{array}
\tag{118}
$$

We leave it to the reader to check that all the required diagrams from Definition 107 commute. ■

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ of (ordinary) categories is <u>*full and faithful*</u> or <u>*fully faithful*</u> if for any two objects x and y in $\mathcal{C}$, the induced map of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(y, x) &\to \mathrm{Hom}_{\mathcal{D}}\big(F(y), F(x)\big) \\
\alpha &\mapsto F(\alpha)
\end{aligned}
\tag{119}
$$

is a bijection. The analogous property for 2-categories and 2-functors is a bit more subtle.

First note that for any two objects x and y of a 2-category $\mathcal{C}$, one can define a *category* $\mathrm{Hom}_{\mathcal{C}}(y, x)$ by setting

$$
\mathrm{Hom}_{\mathcal{C}}(y, x)_0 := \{\alpha \in C_1 \mid s(\alpha) = x \text{ and } t(\alpha) = y\} \tag{120}
$$

and

$$
\mathrm{Hom}_{\mathcal{C}}(y, x)_1 := \{\Sigma \in C_2 \mid ss(\Sigma) = x \text{ and } tt(\Sigma) = y\}. \tag{121}
$$

One can define the source, target, and identity-assigning maps by restricting the ones from $\mathcal{C}$. Composition in $\mathrm{Hom}_{\mathcal{C}}(y, x)$ is the restriction of the vertical composition in $\mathcal{C}$. It is associative and unital by condition (b) of Definition 2.

Definition 122. Let $F : \mathcal{C} \to \mathcal{D}$ be a 2-functor between two 2-categories. F is said to be <u>*fully faithful*</u> if the restriction of F to the induced functor

$$
\mathrm{Hom}_{\mathcal{C}}(y, x) \to \mathrm{Hom}_{\mathcal{D}}\big(F(y), F(x)\big) \tag{123}
$$

is an equivalence of categories, or equivalently, if the above functor on Hom-categories is both essentially surjective and fully faithful, for all objects x and y in $\mathcal{C}$.

Acknowledgements. The author is grateful for the support offered by the NSF (Grant No. 40017-06 05 and 40017-06 06) during which most of these notes were written. The author would also like to thank Aaron Lauda for his guide on Xy-pic as well as its creators.