10/26/2014

Def'n Let Δ be the category whose objects are indexed by the natural numbers and are themselves categories:

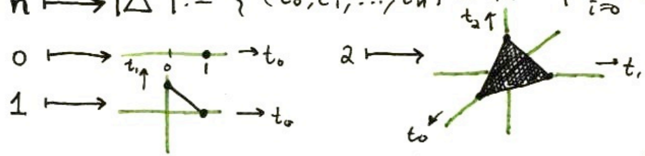
$$\mathbf{n} := (0 \to 1 \to 2 \to \cdots \to n).$$

A morphism from $\mathbf{n}$ to $\mathbf{m}$ is a functor.

Def'n A <u>simplicial set</u> is a presheaf of sets on Δ, i.e. a functor $\Delta^{op} \to \mathrm{Set}$. A morphism of simplicial sets is just a morphism of presheaves i.e. a natural transformation $\Delta^{op} \underset{\Downarrow}{\rightrightarrows} \mathrm{Set}$. Simplicial sets form a category which we denote by sSet.

Before discussing several examples, let's relate this to geometric simplices via a functor $\Delta \to \mathrm{Man}^{L}$, manifolds with corners (we could also take Top). This functor is defined by sending

$$\mathbf{n} \longmapsto |\Delta^n| := \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

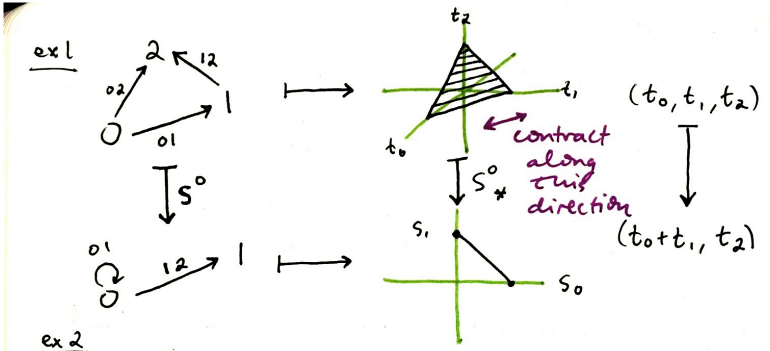


on objects. For a morphism $\mathbf{n} \xrightarrow{\theta} \mathbf{m}$ we define $\theta_*(t_0, \dots, t_n) = (s_0, \dots, s_m)$ where

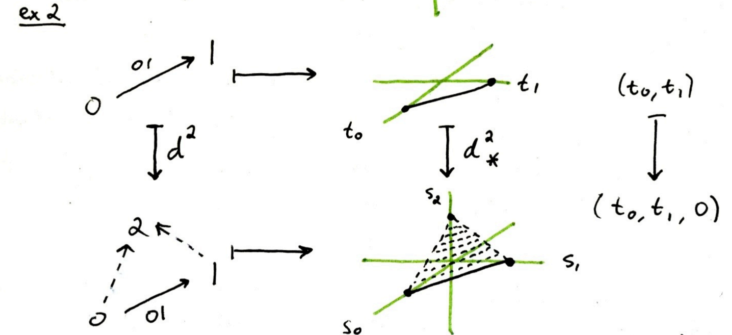
$$s_i = \begin{cases} 0 & \text{when } \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \text{when } \theta^{-1}(i) \neq \emptyset \end{cases}.$$

We illustrate this with two important examples.

ex 1



ex 2



These examples illustrate important types of morphisms in Δ

$$d^i : \mathbf{n-1} \to \mathbf{n} \qquad 0 \leq i \leq n \qquad \text{(cofaces)}$$

$$\& \quad s^j : \mathbf{n+1} \to \mathbf{n} \qquad 0 \leq j \leq n \qquad \text{(codegeneracies)}$$

defined by

$$d^i(0 \to 1 \to \cdots \to i-1 \to i \to i+1 \to \cdots \to n-1)$$

$$:= 0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n-1 \to n \quad \text{(skip } i\text{)}$$

$$\& \quad s^j(0 \to 1 \to \cdots \to j-1 \to j \to j+1 \to \cdots \to n+1)$$

$$:= 0 \to 1 \to \cdots \to j-1 \to j \to j+1 \to \cdots \to n \quad \text{(contract } j\text{)}$$

These functors satisfying the following identities known as the cosimplicial identities

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} && \text{if } i<j\\
s^j d^i &= d^i s^{j-1} && \text{if } i<j\\
s^j d^j &= 1 = s^j d^{j+1} &&\\
s^j d^i &= d^{i-1} s^j && \text{if } i>j+1\\
s^j s^i &= s^i s^{j+1} && \text{if } i\leq j
\end{aligned}$$

If we're given a simplicial set, we have to take contravariance into account. We do this by considering our first example.

<u>ex 1</u> Let $\mathcal{C}$ be a small category. Define $N\mathcal{C} : \Delta^{op} \longrightarrow \text{Set}$ by $N\mathcal{C}(n) := \text{Fun}(n, \mathcal{C})$ functors from n to $\mathcal{C}$. This is called the <u>nerve of $\mathcal{C}$</u>. For the morphisms d^i and s^j from before, we <u>define</u>

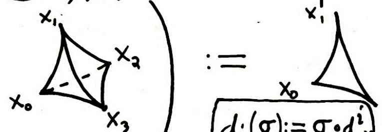
$$\begin{array}{ccc}
\text{Fun}(n,\mathcal{C}) & & \text{Fun}(n,\mathcal{C})\\
\downarrow N\mathcal{C}(d^i) \equiv d_i & \& & \downarrow N\mathcal{C}(s^j) \equiv s_j\\
\text{Fun}(n-1,\mathcal{C}) & & \text{Fun}(n+1,\mathcal{C})
\end{array} \quad \text{by}$$

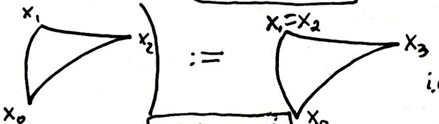
$$d_i(c_0 \to c_1 \to \cdots \to \underbrace{c_{i-1} \to c_i \to c_{i+1}}_{\text{compose}} \to \cdots \to c_n)$$
$$:= c_0 \to c_1 \to \cdots \to c_{i-1} \longrightarrow c_{i+1} \to \cdots \to c_n$$
$$\& \ s_j(c_0 \to c_1 \to \cdots \to c_{j-1} \to c_j \to c_{j+1} \to \cdots \to c_n)$$
$$:= c_0 \to c_1 \to \cdots \to c_{j-1} \to c_j \xrightarrow{1} c_j \to c_{j+1} \to \cdots \to c_n .$$

In general, for a simplicial set $\Delta^{op} \longrightarrow \text{Set}$, the maps d_i are called <u>face maps</u> and s_j <u>degeneracy maps</u>.

<u>ex 2</u> Let X be a topological space. Define $\text{Sing}\,X : \Delta^{op} \longrightarrow \text{Set}$ by $\text{Sing}\,X(n) := \text{Top}(|\Delta^n|, X)$. This is called the singular set of X. For the morphisms d^i and s^j we define

$$\begin{array}{ccc}
\text{Top}(|\Delta^n|, X) & & \text{Top}(|\Delta^n|, X)\\
\downarrow d_i & \& & \downarrow s_j\\
\text{Top}(|\Delta^{n-1}|, X) & & \text{Top}(|\Delta^{n+1}|, X)
\end{array} \quad \text{by}$$

d_i () $:=$ $\boxed{d_i(\sigma) := \sigma \circ d^i_*}$ (i = 2 in picture) i.e. restriction to the face opposite i

& s_j () $:=$ $\boxed{s_j(\sigma) := \sigma \circ s^j_*}$ (j = 1 in picture) i.e. filling a degeneracy at j.

<u>ex 3</u> The standard simplicial n-simplex Δ^n is the simplicial set $\Delta^n : \Delta^{op} \longrightarrow \text{Set}$ defined by $\Delta^n(m) := \hom_\Delta(m, n)$ so that Δ^n is a representable functor. Let's still be explicit about the face and degeneracy maps (they've already been defined). They're given by

$$\begin{array}{c}\Delta^n(m)\\ \downarrow d_i\\ \Delta^n(m-1)\end{array} \left(f : m \to n \right) := \begin{array}{c} m-1 \\ \downarrow d^i \\ m \xrightarrow[f]{} n \end{array} \quad \&$$

$$\begin{array}{c}\Delta^n(m)\\ \downarrow s_j\\ \Delta^n(m+1)\end{array} \left(f : m \to n \right) := \begin{array}{c} m+1 \\ \downarrow s^j \\ m \xrightarrow[f]{} n \end{array} .$$

Before moving on, we notice that our previous example is nothing more than the Yoneda embedding $\Delta \hookrightarrow Set^{\Delta^{op}} \equiv sSet$.

Claim Simplicial set maps $f : \Delta^n \to \Delta^m$ are in natural one-to-one correspondence with morphisms $f : n \to m$ in Δ.

Yoneda Lemma Let c be an object of a category $\mathcal{C}$ and $F : \mathcal{C}^{op} \to Set$ a presheaf of sets on $\mathcal{C}$. Then the function $Nat(hom_{\mathcal{C}}(\cdot, c), F) \longrightarrow Fc$ defined by $\eta \longmapsto \eta_c(1_c)$ is a natural bijection.

Corollary Let $Y : \Delta^{op} \to Set$ be a simplicial set. Then maps $\Delta^n \to Y$ are in one-to-one correspondence with elements of Y_n. Such elements are called **n-simplices**.

Defn The **standard n-simplex** is the element $\iota_n \in \Delta^n_n$ corresponding to $id_{\Delta^n} : \Delta^n \to \Delta^n$ under the Yoneda bijection from above.

Thus, any n-simplex $\sigma \in Y_n$ is the image of the standard n-simplex ι_n under a unique classifying map $\Delta^n \xrightarrow{\Sigma} Y$: $\Sigma_n(\iota_n) = \sigma$.

Furthermore, the coface and codegeneracy maps $d^i : n-1 \to n$ and $s^j : n+1 \to n$ define unique maps $d^i : \Delta^{n-1} \to \Delta^n$ and $s^j : \Delta^{n+1} \to \Delta^n$ respectively (we use the same notation sometimes).

Defn Let X and Y be two simplicial sets and let $f : X \to Y$ be a map of simplicial sets. The image of f, denoted by $im(f)$ is the simplicial set defined by $im(f)(n) := im(f_n)$. The face and degeneracy maps are those on Y restricted to $im(f)$. The reason this is well-defined is by naturality of f.

More explicitly, take for instance

$$\begin{array}{ccc} X_n & \xrightarrow{d_i^X} & X_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ Y_n & \xrightarrow[d_i^Y]{} & Y_{n-1} \end{array}$$

If $\sigma \in im(f_n)$ then $d_i^Y(\sigma)$ is in the image of f_{n-1} by naturality.

Ex/Defn The **i^{th} face of Δ^n** is the simplicial subset $\partial_i \Delta^n$ of Δ^n defined by $\partial_i \Delta^n := im(d^i : \Delta^{n-1} \to \Delta^n)$.

Defn Let X and Y be two simplicial subsets of a simplicial set Z (this means $X_n \subset Z_n$ and $Y_n \subset Z_n$ for all n and all face and degeneracy maps are respected). Then the union $X \cup Y$ is the simplicial set defined by $(X \cup Y)(n) := X_n \cup Y_n$. The face and degeneracy maps are those of Z restricted to subsets.

Finally, after these numerous definitions and examples we come to our fourth and most important example of a simplicial set, the k^{th} horn.

ex 4 The k^{th} horn of Δ^n is the simplicial set Λ^n_k defined by the union

$$\Lambda^n_k := \bigcup_{i \neq k} \partial_i \Delta^n$$

of all the i^{th} faces of Δ^n except the k^{th} face.

ex 5 The simplicial n-sphere $\partial\Delta^n$ is the simplicial subset of Δ^n defined by

$$\partial\Delta^n := \bigcup_{i} \partial_i \Delta^n .$$

Defn Let X be a simplicial set. The simplex category $\Delta \downarrow X$ of X has objects the simplices of X, i.e. simplicial set maps $\Delta^n \xrightarrow{\sigma} X$ over all n. A morphism from $\Delta^n \xrightarrow{\sigma} X$ to $\Delta^m \xrightarrow{\tau} X$ is a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[d, "\theta"'] \arrow[dr, "\sigma"] & \\ \Delta^m \arrow[r, "\tau"'] & X \end{tikzcd} .$$

Note that by Yoneda, we can simply think of objects as elements $\sigma \in X_n$ and morphisms $\theta : \sigma \to \tau$ as maps $\theta : n \to m$ such that $X(\theta)(\tau) = \sigma$.

The Density Theorem There is an isomorphism of simplicial sets (in sSet)

$$X \cong \operatorname*{colim}_{n \in \mathbb{N},\, \sigma \in \Delta \downarrow X} \Delta^n .$$

This says that a simplicial set is determined by its simplices.

Defn Let X be a simplicial set. The geometric realization $|X|$ of X is the topological space defined by the colimit

$$|X| := \operatorname*{colim}_{n \in \mathbb{N},\, \sigma \in \Delta \downarrow X} |\Delta^n| .$$

ex1 We now have an abuse of notation. On the one hand, we've defined $|\Delta^n|$ as the topological standard n-simplex in $\mathbb{R}^{n+1}$. On the other hand, we have $|\Delta^n|$ the geometric realization of the standard simplicial n-simplex Δ^n. The proof that the two are homeomorphic, in fact diffeomorphic in Man^L, is easy to see if you just think about it. The realization is just

$$|\Delta^n|_{\text{Realiz}} := \operatorname*{colim}_{m \in \mathbb{N},\, \sigma \in \Delta \downarrow \Delta^n} |\Delta^m|_{\text{topological}}$$

but all of these maps just embed lower-dimensional simplices and squash higher-dimensional simplices in the standard n-simplex $\imath_n$. Thus,

$$\operatorname*{colim}_{m \in \mathbb{N},\, \sigma \in \Delta \downarrow X} |\Delta^m| = \operatorname*{colim}_{\imath_n} |\Delta^n| = |\Delta^n|$$ as needed!

ex2 $|\partial_i \Delta^n| = \operatorname*{colim}_{m \in \mathbb{N},\, \sigma \in \Delta \downarrow \partial_i \Delta^n} |\Delta^m| = \operatorname*{colim}_{d^i(\imath_{n-1})} |\Delta^{n-1}| = |\Delta^{n-1}|$

viewed as the i^{th} face in $|\Delta^n|$.

ex 3 $|\Lambda^n_k| = \operatorname*{colim}_{m \in \mathbb{N},\, \sigma \in \Delta \downarrow \Lambda^n_k} |\Delta^m| = \operatorname{colim} \left(\begin{array}{l} \text{all faces of } \Delta^n \\ \text{except the } k^{th} \\ \text{face with appropr.} \\ \text{gluings} \end{array} \right)$

Remember, colimit takes the disjoint union and then glues using all the morphisms

Claim Both $\mathrm{Sing}: \mathrm{Top} \to \mathrm{sSet}$ and $|\cdot| : \mathrm{sSet} \to \mathrm{Top}$ define functors.

Proposition $|\cdot|$ is left-adjoint to Sing, i.e. $\hom_{\mathrm{Top}}(|X|, Y) \cong \hom_{\mathrm{sSet}}(X, \mathrm{Sing}\, Y)$ natural in X and Y.

proof $\hom_{\mathrm{Top}}(|X|, Y) = \hom_{\mathrm{Top}}\left(\operatorname{colim}_{\sigma:\Delta^n \to X} |\Delta^n|, Y\right)$

$$\cong \lim_{\sigma:\Delta^n \to X} \hom_{\mathrm{Top}}(|\Delta^n|, Y)$$

because contravariant representable functors take colimits to limits (the representable functor here is $\hom_{\mathrm{Top}}(\cdot, Y)$)

$$\cong \lim_{\sigma:\Delta^n \to X} \hom_{\mathrm{sSet}}(\Delta^n, \mathrm{Sing}\, Y)$$

by the Yoneda Lemma and the def'n of Sing

$$\cong \hom_{\mathrm{sSet}}\left(\operatorname{colim}_{\sigma:\Delta^n \to X} \Delta^n, \mathrm{Sing}\, Y\right)$$

again by the property of contravariant representable functors

$$\cong \hom_{\mathrm{sSet}}(X, \mathrm{Sing}\, Y)$$

by the density theorem. ■

Proposition Let X be a simplicial set. Then $|X|$ is a CW-complex.

Def'n A topological space Y is a compactly generated Hausdorff space if Y is Hausdorff and if every compactly closed subspace (meaning its intersection with every compact subset of Y is closed) is itself closed.

Fact CW-complexes are compactly generated Hausdorff spaces.

Fact compactly generated Hausdorff spaces with ordinary continuous maps as morphisms form a category CGHaus.

Corollary Geometric realization factors through CGHaus, i.e. $|\cdot| : \mathrm{sSet} \to \mathrm{CGHaus}$

Side remark $|\cdot| : \mathrm{sSet} \to \mathrm{CGHaus}$ preserves finite limits. In particular, $|X \times Y| \cong |X| \times_{ke} |Y|$ where $\times_{ke}$ is the product in CGHaus (and does not equal the product in Top). This is to be contrasted with the fact that $|X \times Y| \ncong |X| \times |Y|$ in Top.

We now switch gears a bit and discuss model categories since we have two examples already set up.

Def'n A closed model category $\mathcal{C}$ is a category $\mathcal{C}$ together with three classes of morphisms called

i) fibrations ii) cofibrations & iii) weak equivalences satisfying the following axioms.

CM1: $\mathcal{C}$ is closed under all finite limits and colimits.

CM2: Suppose the diagram
$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$
commutes in $\mathcal{C}$. If any two of f, g, and h are weak equivalences, then so is the third.

CM3: If f is a retract of g and g is a weak equivalence, fibration, or cofibration, then so is f.

CM4: For any diagram of the form
$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & Y \end{tikzcd}$$
where i is a cofibration, p is a fibration, and either i or p is a weak equivalence then there exists a map $\theta: V \to X$ making the diagram on the left commute.
$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed, "\theta"] & Y \end{tikzcd}$$

CM5: Any morphism $f: X \to Y$ can be factored in two ways as

(a) $f = p \circ i$, where p is a fibration and i is both a cofibration and a weak equivalence

(b) $f = q \circ j$, where q is both a fibration and a weak equivalence and j is a cofibration.

Theorem Let the fibrations in CGHaus be Serre fibrations (recall that a map $f: X \to Y$ is a Serre fibration if for any CW-complex Z equipped with maps
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ Z \times [0,1] \arrow[r, "h"'] & Y \end{tikzcd}$$
so that this diagram commutes, then there exists a map $Z \times [0,1] \xrightarrow{\tilde{h}} X$ making the diagram
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ Z \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd}$$
commute).

The weak equivalences are defined to be the weak homotopy equivalences (isomorphisms on homotopy groups).

Now, by a fact of model categories, the specification of cofibrations is determined by CM4. More precisely, a morphism $U \xrightarrow{i} V$ is a cofibration if for any diagram
$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & Y \end{tikzcd}$$
where p is both a fibration and a weak equivalence then there exists a map $V \xrightarrow{\theta} X$ so that
$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed, "\theta"] & Y \end{tikzcd}$$
commutes.

Then CGHaus with these fibrations, cofibrations, and weak equivalences defines a model category.

Our next example will be sSet but we need to define fibrations, cofibrations, and weak equivalences. This will take some time.

Defn A map $f: X \to Y$ of simplicial sets is a <u>Kan fibration</u> if for every commutative diagram of simplicial sets
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$
there is a simplicial map $\Theta: \Delta^n \to X$ making
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\Theta"] & Y \end{tikzcd}$$
commute.

<u>Remark</u>: This looks a lot like a Serre fibration because $|\Lambda^n_k| \times [0,1] \cong |\Delta^n|$.

Defn A <u>Kan complex / ∞-groupoid</u> is a fibrant simplicial set Y, meaning the map $Y \longrightarrow * \equiv \Delta^0$ is a Kan fibration.

Equivalently, Y is a Kan complex if and only if every map $\Lambda^n_k \xrightarrow{\alpha} Y$ can be extended to a map $\Delta^n \to Y$ so that

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

commutes.

ex1 $\mathrm{Sing}\,X$ is a Kan complex for any topological space X.

proof Let $\Lambda^n_k \xrightarrow{\alpha} \mathrm{Sing}\,X$ be a map. By the adjunction between $|\cdot|$ & Sing this corresponds to a map $|\Lambda^n_k| \xrightarrow{\tilde{\alpha}} X$. Since $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$, an extension $|\Delta^n| \xrightarrow{a} X$ so that

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed, "a"'] & \end{tikzcd}$$

commutes exists. Again by adjunction this extension corresponds to $\Delta^n \xrightarrow{\tilde{a}} \mathrm{Sing}\,X$. Thus

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & \mathrm{Sing}\,X \\ \Delta^n \arrow[ur, "\tilde{a}"'] & \end{tikzcd}$$

is the required diagram. ■

ex2 Let G be a group and let $\mathcal{G}$ be the associated category with one object all of whose morphisms are just elements of G. Then $N\mathcal{G}$ is a Kan complex. In fact, $N\mathcal{C}$, for any small groupoid $\mathcal{C}$, is a Kan complex.

proof Let $\Lambda^n_k \xrightarrow{\alpha} N\mathcal{C}$ be a map of simplicial sets. Such a map is completely determined by the images of $d^i(\iota_{n-1})$ for all i with $0 \le i \le n$ and $i \neq k$ in the following sense. First, by image of $d^i(\iota_{n-1})$, we mean to look at $\Lambda^n_k(n-1) \xrightarrow{\alpha_{n-1}} N\mathcal{C}(n-1)$ and specifically the images $\alpha_{n-1}(d^i(\iota_{n-1}))$. This makes sense because by definition $\Lambda^n_k = \bigcup_{i\neq k} \partial_i \Delta^n = \bigcup_{i \neq k} \mathrm{im}(d^i : \Delta^{n-1} \to \Delta^n)$ and when we plug in m we have $\Lambda^n_k(m) = \bigcup_{i\neq k} \mathrm{im}(d^i(m) : \hom(m,n-1) \to \hom(m,n))$ which is defined by

$$\hom(m,n-1) \longrightarrow \hom(m,n)$$

$$\begin{tikzcd} m \arrow[d, "f"] & \longmapsto & m \arrow[d, "f"] \arrow[dr, "d^i \circ f"] & \\ n-1 & & n-1 \arrow[r, "d^i"'] & n \end{tikzcd}$$

and in the particular case of $m = n-1$ and $f = id_{n-1}$ we have

$$\begin{tikzcd} n-1 \arrow[d, "id_{n-1}"] & \longmapsto & n-1 \arrow[d, "id_{n-1}"] \arrow[dr, "d^i \circ id_{n-1} \equiv d^i(\iota_n)"] & \\ n-1 & & n-1 \arrow[r] & n \end{tikzcd}$$

In fact, these elements d^i all correspond to the ones determined by the classifying maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] & N\mathcal{C} \\ \Delta^{n-1} \arrow[u, "d^i"] \arrow[ur, "\alpha \circ d^i"'] & \end{tikzcd}$$

for all $i \neq k$. So, the claim is that $\Lambda^n_k \xrightarrow{\alpha} N\mathcal{C}$ is determined by these n maps d^i for $0 \le i \le n$ and $i \neq k$, i.e. α is uniquely defined once $\alpha(d^i(\iota_{n-1})) \in N\mathcal{C}(n-1)$ is specified for all i as above. We leave the proof to the reader. Let's instead describe what this gives us explicitly for low values of n and k.

$n=1, k=0$: $\Lambda^1_0 \longrightarrow N\mathcal{C}$

we only have

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] & N\mathcal{C} \\ \Delta^0 \cong * \arrow[u, "d^1"] \arrow[ur] & \end{tikzcd}$$

which just specifies an object of $\mathcal{C}$. In this case,

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d, hook] & N\mathcal{C} \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

the extension is merely the identity morphism at that object.

$n=1, k=1$: is similar

$n=2, k=0$: $\Lambda^2_0 \to N\mathcal{C}$ is specified by two maps

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] & N\mathcal{C} \\ \Delta^1 \arrow[u, shift left, "d^1"] \arrow[u, shift right, "d^2"'] \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

which specify two morphisms in $\mathcal{C}$ as in

$$\begin{tikzcd} & c_1 & \\ c_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & c_2 \end{tikzcd}$$

In this case

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & N\mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

is given by the pair of composable morphisms

$$c_0 \xrightarrow{f} c_1 \xrightarrow{g \circ f^{-1}} c_2,$$

which is allowed because $\mathcal{C}$ is a groupoid.

$n=2, k=1$: In this case we have

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] & N\mathcal{C} \\ \Delta^1 \arrow[u, shift left, "d^0"] \arrow[u, shift right, "d^2"'] \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

which specify the two morphisms

$$\begin{tikzcd} & c_1 \arrow[dr, "h"] & \\ c_0 \arrow[ur, "f"] & & c_2 \end{tikzcd}$$

Here

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & N\mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

is just what we see, namely the pair of composable morphisms.

$n=2, k=2$: Here we have $\Lambda^2_2 \to N\mathcal{C}$ which specifies two morphisms as in

$$\begin{tikzcd} c_1 \arrow[dr, "h"] & \\ c_0 \arrow[r, "g"'] & c_2 \end{tikzcd}$$

The map

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d, hook] & N\mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

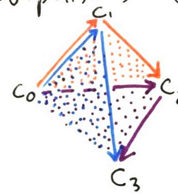
is specied by the pair of composable morphisms

$$\begin{tikzcd} & c_1 \arrow[dr, "h"] & \\ c_0 \arrow[ur, "h^{-1} \circ g"] & & c_2 \end{tikzcd}$$

$n=3, k=0$: $\Lambda^3_0 \to N\mathcal{C}$ means we have three maps

$$\begin{tikzcd} \Lambda^3_0 \arrow[r] & N\mathcal{C} \\ \Delta^2 \arrow[u, "d^1"] \arrow[u, "d^2"] \arrow[u, "d^3"] \arrow[ur] \arrow[ur] \arrow[ur] & \end{tikzcd}$$

all of which look like three pairs of composable morphisms (drawn in different colors)



and the required map

$$\begin{tikzcd} \Lambda^3_0 \arrow[r] \arrow[d, hook] & N\mathcal{C} \\ \Delta^3 \arrow[ur, dashed] & \end{tikzcd}$$

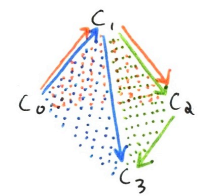
is a triple of composable morphisms given by

$$\begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur] & & c_2 \arrow[dl] \\ & c_3 & \end{tikzcd}$$

$n=3, k=1$:

$$\begin{tikzcd} \Lambda^3_0 \arrow[r] & N\mathcal{C} \\ \Delta^2 \arrow[u] \arrow[u] \arrow[u] \arrow[ur] \arrow[ur] \arrow[ur] & \end{tikzcd}$$

is given by



while

$$\begin{tikzcd} \Lambda^3_0 \arrow[r] \arrow[d, hook] & N\mathcal{C} \\ \Delta^3 \arrow[ur, dashed] & \end{tikzcd}$$

is given by

$$\begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur] & & c_2 \arrow[dl] \\ & c_3 & \end{tikzcd}$$

again.

In fact all the higher maps are simple like this. The only nontrivial maps were Λ^2_0 and Λ^2_2 to check the Kan condition.

<u>Warning</u>: The previous example has a potentially flawed argument. I seem to be misunderstanding something subtle — see Goerss & Jardine pg 14.

Anyway, our next goal is to define the weak equivalences in sSet. If we wanted to, we could define this in sSet without using $|\cdot| : \mathrm{sSet} \to \mathrm{CGHaus}$ but then we could only define a weak equivalence of Kan complexes as opposed to general simplicial sets. This is because simplicial homotopy is not an equivalence relation for general simplicial sets (see pg 25) but it is for Kan complexes. Remember, to define homotopy groups of spaces, we have to mod out by homotopy. A similar thing happens here but if we don't have an equivalence relation we can't mod out. Anyway, we can bipass this issue by using the following definition.

Def'n A map $f : X \to Y$ is a <u>(simplicial) weak equivalence</u> iff the induced realization map $|f| : |X| \to |Y|$ is a weak equivalence in CGHaus.

Def'n A <u>cofibration of simplicial sets</u> is an inclusion map.

Theorem The category of simplicial sets equipped with the above definitions of Kan fibrations, weak equivalences, and cofibrations is a closed model category.

We now relate CGHaus and sSets together with their model structures.

For the next few words, our reference will be Freed-Hopkins "Chern-Weil forms and abstract homotopy theory" (together with Goerss and Jardine of course).

Def'n Let $\mathcal{C}$ be a model category and $\mathcal{D}$ an ordinary category. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an <u>invariant of weak equivalence</u> if it sends weak equivalences to isomorphisms.

Def'n Let $\mathcal{C}$ be as above and denote the weak equivalences by $\mathcal{W}$. The <u>localization of $\mathcal{C}$ with respect to $\mathcal{W}$</u> is a category $\mathrm{ho}\mathcal{C}$, called the homotopy category of $\mathcal{C}$, together with a functor $L : \mathcal{C} \to \mathrm{ho}\mathcal{C}$ that is an invariant of weak equivalence such that for any other invariant of weak equivalences $F : \mathcal{C} \to \mathcal{D}$ there exists a unique functor $\mathrm{ho}\mathcal{C} \to \mathcal{D}$ making the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L"] \arrow[dr, "F"'] & \mathrm{ho}\mathcal{C} \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd}$$

commute.

Idea: $\mathrm{ho}\mathcal{C}$ is constructed by inverting weak equivalences.

Imprecise claim: Localization is functorial. In particular, it can be applied to a morphism $F : \mathcal{C} \to \mathcal{D}$, of model categories.

Theorem The functors

$$\begin{tikzcd} \mathrm{ho}(\mathrm{sSet}) \arrow[r, shift left, "|\cdot|"] & \mathrm{ho}(\mathrm{Top}) \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}$$

form an equivalence of homotopy categories.