Noncommutative disintegration

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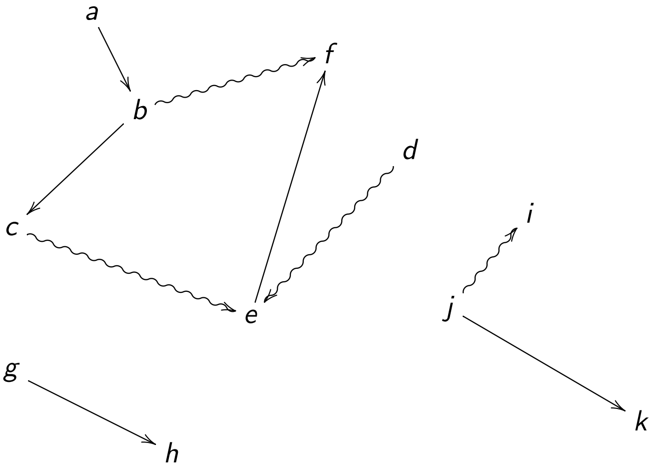
Category theory as a theory of processes

Processes can be deterministic or non-deterministic



Stochastic maps

Let X and Y be finite sets. A *stochastic map* $r : Y \rightsquigarrow X$ assigns a probability measure on X to every point in Y. It is a function whose value at a point "spreads out" over the codomain.

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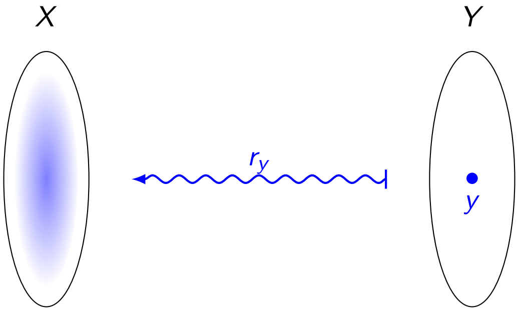

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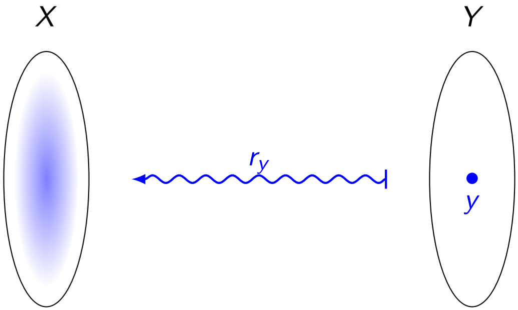
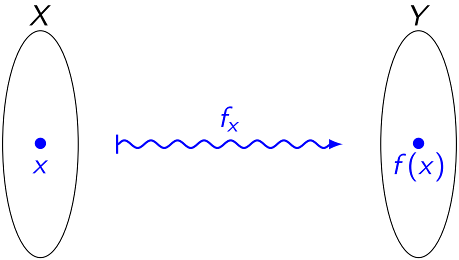


The value $r_y(x)$ of r_y at x is denoted by r_{xy}. Since r_y is a probability measure, $r_{xy} \geq 0$ for all x and y. Also, $\sum_{x \in X} r_{xy} = 1$ for all y.

Stochastic maps from functions

A function $f : X \to Y$ induces a stochastic map $f : X \rightsquigarrow Y$ via

$$f_{yx} := \delta_{yf(x)}$$



where $\delta_{yy'}$ is the Kronecker delta and equals 1 if and only if $y = y'$ and is zero otherwise.

Composing stochastic maps

The *composition* $\nu \circ \mu : X \rightsquigarrow Z$ of $\mu : X \rightsquigarrow Y$ followed by $\nu : Y \rightsquigarrow Z$ is defined by matrix multiplication

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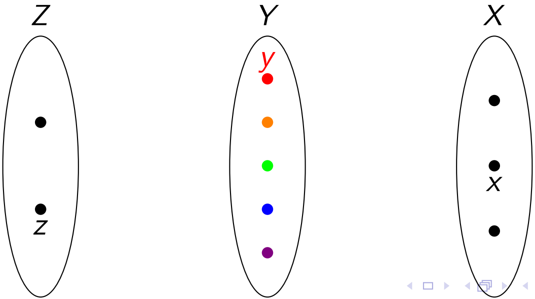

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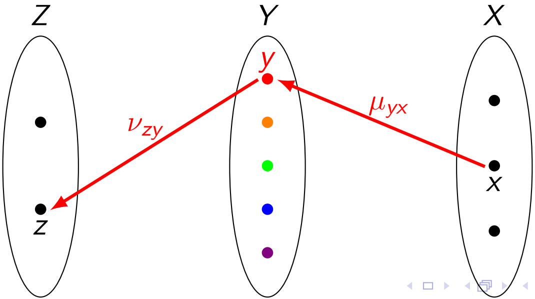

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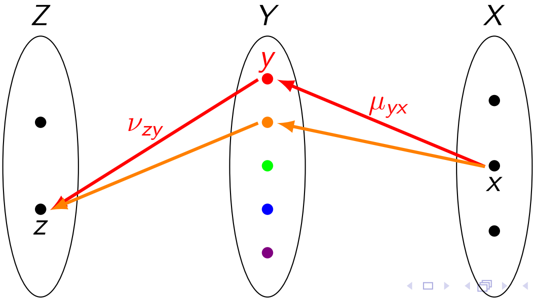

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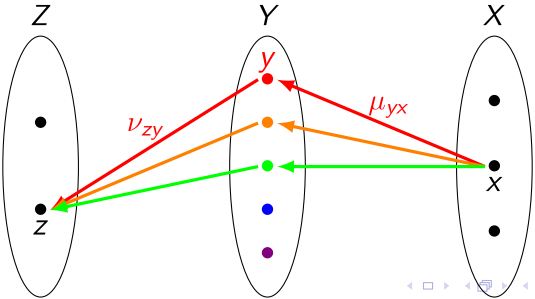

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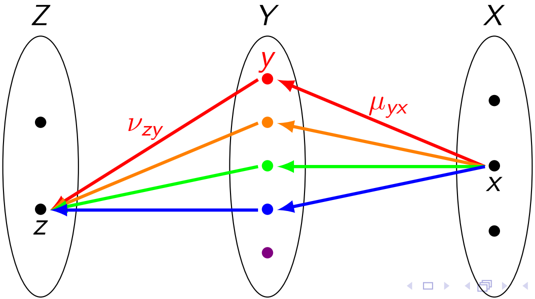

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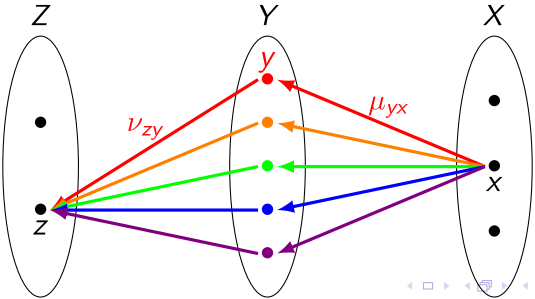

Special case: probability measures

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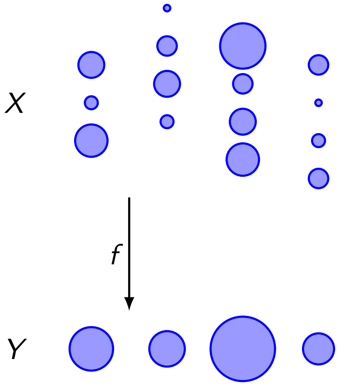
- If $f : X \to Y$ is a function, the composition $f \circ \mu : \{\bullet\} \rightsquigarrow Y$ is the pushforward of μ along f.
- If $f : X \rightsquigarrow Y$ is a stochastic map, the composition $f \circ \mu : \{\bullet\} \rightsquigarrow Y$ is a generalization of the pushforward of a measure. The measure $f \circ \mu$ on Y is given by $(f \circ \mu)(y) = \sum_{x \in X} f_{yx}\mu(x)$ for each $y \in Y$.

Stochastic maps and their compositions form a category

Composition of stochastic maps is associative and the identity function on any set acts as the identity morphism.

Disintegrations as a section

Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. f combines the water droplets and their volume (probabilities) add when they combine under f.



Disintegrations as a section

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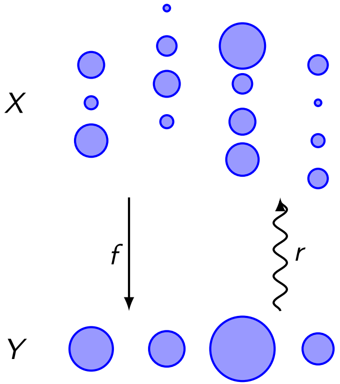
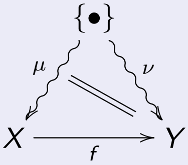


Disintegrations: diagrammatic definition

Definition

Let (X, μ) and (Y, ν) be probability spaces and let $f : X \to Y$ be a function such that the diagram on the right commutes.



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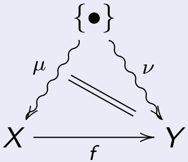
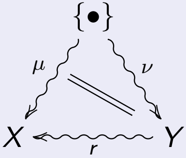


A *disintegration* of μ over ν consistent with f is a stochastic map $r : Y \rightsquigarrow X$ such that



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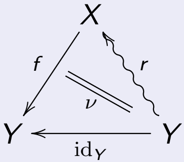

the latter diagram signifying commutativity ν-a.e.

Classical disintegrations exist and are unique a.e.

Theorem

Let (X, μ) and (Y, ν) be finite sets equipped with probability measures μ and ν. Let $f : X \to Y$ be a measure-preserving function. Then there exists a unique (ν-a.e.) disintegration $r : Y \rightsquigarrow X$ of μ over ν consistent with f.

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In fact, a formula for the disintegration is

$$r_{xy} := \begin{cases} \mu_x \delta_{yf(x)} / \nu_y & \text{if } \nu_y > 0 \\ 1/|X| & \text{otherwise} \end{cases}$$

Matrix algebras

- Let $\mathcal{M}_n(\mathbb{C})$ denote the set of complex $n \times n$ matrices. It is an example of a C^*-algebra: we can add and multiply $n \times n$ matrices, the operator norm gives a norm, and A^* is the conjugate transpose of A.

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- In particular, $\mathbb{C}^X$, functions from a finite set X to $\mathbb{C}$, is a *commutative* C^*-algebra. A basis for this algebra as a vector space is $\{e_x\}_{x \in X}$ defined by $e_x(x') := \delta_{xx'}$.

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- If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$ can be viewed as $n \times n$ matrices with entries in $\mathcal{A}$. It has a natural C^*-algebra structure.

Completely positive maps and $*$-homomorphisms

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- If $\omega : \mathcal{M}_n(\mathbb{C}) \rightsquigarrow \mathbb{C}$ is a state, there exists a unique $n \times n$ positive matrix ρ such that $\mathrm{tr}(\rho) = 1$ and $\mathrm{tr}(\rho A) = \omega(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$.

From finite sets to finite-dimensional C^*-algebras I

There is a (contravariant) functor from finite sets and stochastic maps to finite-dimensional C^*-algebras and completely positive maps.

category theory	classical/ commutative	quantum/ noncommutative	physics/ interpretation
object	set	C^*-algebra	phase space observables
$\to$ morphism	function	$*$-homomorphism	deterministic process
$\rightsquigarrow$ morphism	stochastic map	completely positive map	non-deterministic process
monoidal product	cartesian product $\times$	tensor product $\otimes$	combining systems
$\rightsquigarrow$ to/from monoidal unit	probability measure	C^*-algebra state/ density matrix	physical state

From finite sets to finite-dimensional C^*-algebras II

Briefly, this functor is given by

$$X \mapsto \mathbb{C}^X$$
$$\big(f : X \rightsquigarrow Y\big) \mapsto \Big(\mathbb{C}^Y \ni e_y \mapsto \sum_{x \in X} f_{yx} e_x \in \mathbb{C}^X\Big)$$

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Therefore, an arbitrary function $\varphi = \sum_{y \in Y} \varphi(y) e_y \in \mathbb{C}^Y$ gets sent to

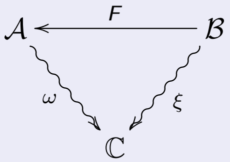
$$\sum_{y \in Y} \varphi(y) \sum_{x \in X} f_{yx} e_x = \sum_{y \in Y} \varphi(y) \sum_{x \in f^{-1}(y)} e_x = \sum_{x \in X} \varphi(f(x)) e_x = \varphi \circ f$$

the pullback of φ along f.

Non-commutative disintegrations

Definition (P-Russo)

Let $(\mathcal{A}, \omega)$ and $(\mathcal{B}, \xi)$ be C^*-algebras equipped with states. Let $F : \mathcal{B} \to \mathcal{A}$ be a $*$-homomorphism such that the diagram on the right commutes.



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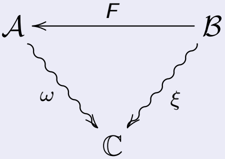
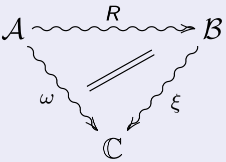


A *disintegration* of ω over ξ consistent with F is a completely positive map $R : \mathcal{A} \rightsquigarrow \mathcal{B}$ such that



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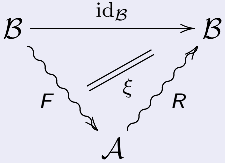

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Fix $n, p \in \mathbb{N}$. *Let*

$$\begin{array}{ccc} \mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{\;F\;} & \mathcal{M}_n(\mathbb{C}) \\ & {\scriptstyle \mathrm{tr}(\rho\,\cdot\,)\equiv\omega} \searrow \quad \swarrow {\scriptstyle \xi\equiv\mathrm{tr}(\sigma\,\cdot\,)} & \\ & \mathbb{C} & \end{array}$$

be a commutative diagram with F *the* $*$*-homomorphism given by the block diagonal inclusion* $F(A) = \mathrm{diag}(A, \ldots, A)$.

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$$\begin{array}{ccc} \mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{\;F\;} & \mathcal{M}_n(\mathbb{C}) \\ & {\scriptstyle \mathrm{tr}(\rho\,\cdot\,)\equiv\omega} \searrow \quad \swarrow {\scriptstyle \xi\equiv\mathrm{tr}(\sigma\,\cdot\,)} & \\ & \mathbb{C} & \end{array}$$

be a commutative diagram with F *the* $*$*-homomorphism given by the block diagonal inclusion* $F(A) = \mathrm{diag}(A, \ldots, A)$. *A disintegration of* ω *over* ξ *consistent with* F *exists* **if and only if** *there exists a density matrix* $\tau \in \mathcal{M}_p(\mathbb{C})$ *such that* $\rho = \tau \otimes \sigma$.

Example 1: Einstein-Podolsky-Rosen

Theorem (P-Russo)

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \& \qquad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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Proof.

ρ is entangled (not separable) and therefore cannot be expressed as the tensor product of any two 2×2 density matrices. □

Example 2: Diagonal density matrices

Theorem (P-Russo)

Fix $p_1, p_2, p_3, p_4 \geq 0$ *with* $p_1 + p_2 + p_3 + p_4 = 1$, $p_1 + p_3 > 0$, *and* $p_2 + p_4 > 0$. *Let*

$$\rho = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \qquad \& \qquad \sigma = \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix}$$

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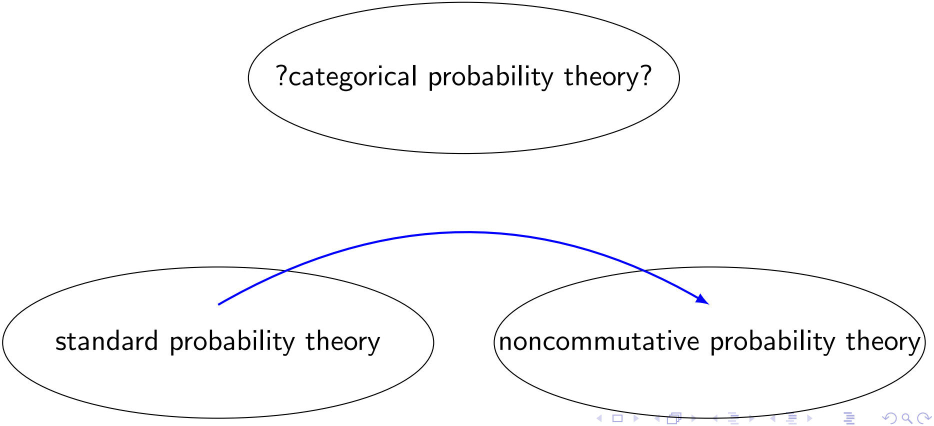
$$p_1 p_4 = p_2 p_3.$$

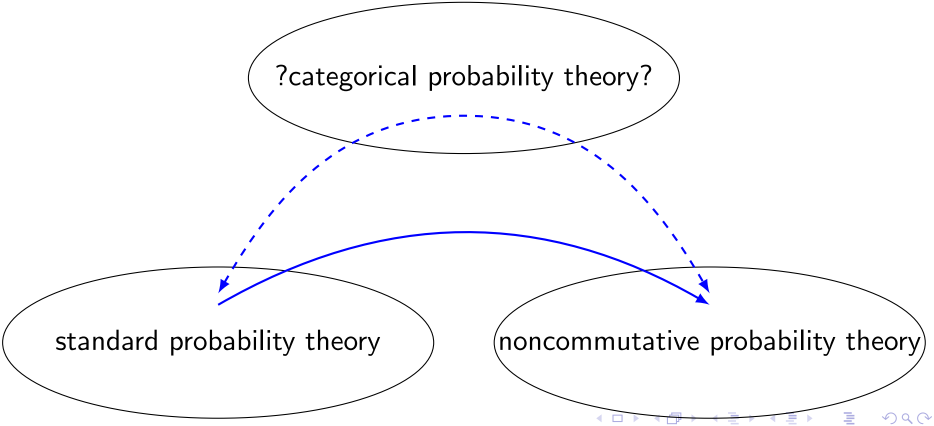
Summary

Formulating concepts in probability theory categorically enables one to abstract these concepts to contexts beyond their initial domain.



Summary

Formulating concepts in probability theory categorically enables one to abstract these concepts to contexts beyond their initial domain. However, we still lack a full categorical probability theory. Amazing discoveries are yet to be made!



Thank you!

Thank you for your attention!