Noncommutative disintegration

Arthur J. Parzygnat* & Benjamin P. Russo†

*University of Connecticut
†Farmingdale State College SUNY
NEAM 2018 SUNY New Paltz

October 19, 2018

1. Deterministic and nondeterministic processes

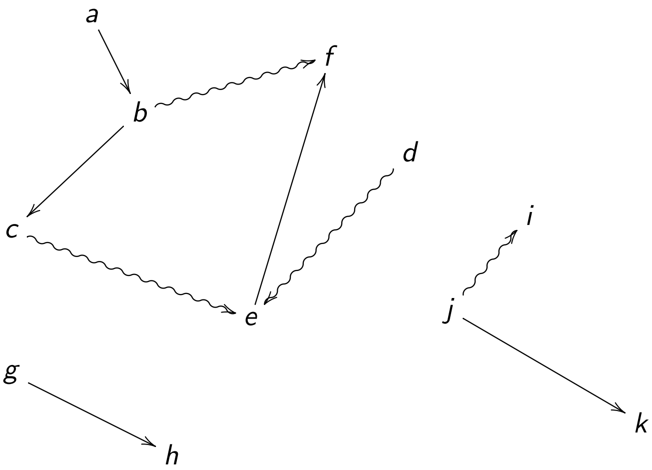
2. Transition kernels
   - Standard definitions
   - The category of transition kernels

3. Classical disintegrations
   - Classical disintegrations: intuition
   - Diagrammatic disintegrations
   - Existence and uniqueness of disintegrations

4. Quantum disintegrations
   - Completely positive maps and $*$-homomorphisms
   - Non-commutative disintegrations
   - Existence and uniqueness
   - Examples

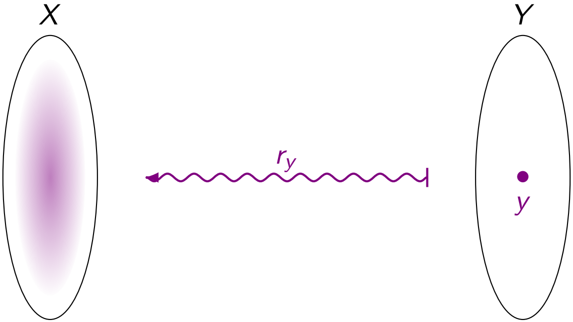
Category theory as a theory of processes

Processes can be deterministic or non-deterministic



Transition kernels

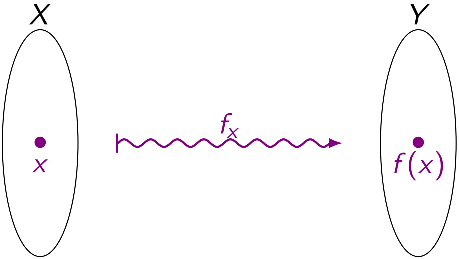
A *transition kernel* assigns a measure on X to every point in Y. It is a function whose value at a point "spreads out" over the codomain.



Transition kernels from measurable functions

If (X, Σ) and (Y, Ω) are measurable spaces,
a measurable function $f : X \to Y$ induces a transition kernel $f : X \rightsquigarrow Y$ via

$$f_x(E) := \chi_E\big(f(x)\big)$$

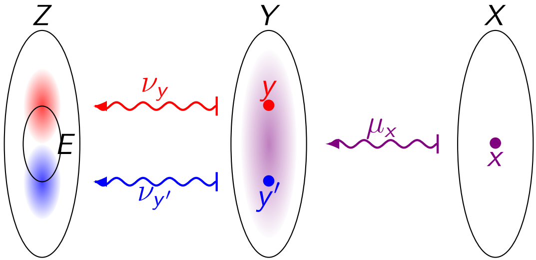


where χ_E is the indicator function on a measurable subset E.

Composing transition kernels

The *composition* $\nu \circ \mu : X \rightsquigarrow Z$ of $\mu : X \rightsquigarrow Y$ followed by $\nu : Y \rightsquigarrow Z$ is defined by

$$(\nu \circ \mu)_x(E) := \int_Y \nu_y(E)\, d\mu_x(y).$$



Special case I: finite sets

If X, Y, and Z finite sets,

$$(\nu \circ \mu)_x(E) := \int_Y \nu_y(E)\, d\mu_x(y).$$

becomes matrix multiplication

$$(\nu \circ \mu)_x(z) = \sum_{y \in Y} \nu_{zy}\mu_{yx}.$$

where $\mu_{yx} := \mu_x(\{y\})$.

Special case II: measures

- A measure μ on X can be viewed as a transition kernel $\mu : \{\bullet\} \rightsquigarrow X$ from a single element set.

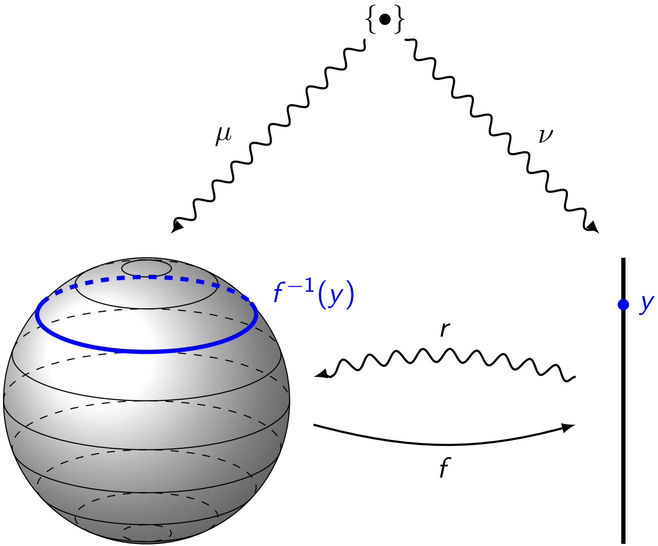
Special case II: measures

- A measure μ on X can be viewed as a transition kernel $\mu : \{\bullet\} \rightsquigarrow X$ from a single element set.
- If $f : X \to Y$ is a measurable function, the composition $\mu \circ f : \{\bullet\} \rightsquigarrow Y$ is the pushforward of μ along f.

Transition kernels and their compositions form a category

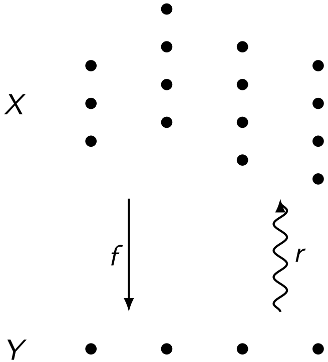
Composition of transition kernels is associative and the identity function on any measurable space acts as the identity morphism.

Disintegrations: localizing measures on slices



Disintegrations: as a section

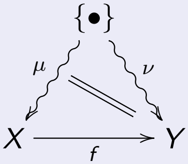
A disintegration is a "stochastic" section



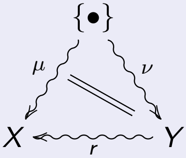
Disintegrations: diagrammatic definition

Definition

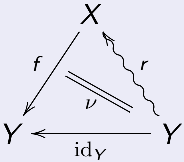
Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces and let $f : X \to Y$ be a measurable map such that the diagram on the right commutes.



A *disintegration* of μ over ν consistent with f is a transition kernel $r : Y \rightsquigarrow X$ such that



and



the latter diagram signifying commutativity ν-a.e.

Existence and uniqueness of disintegrations

Theorem

Let (X, μ) and (Y, ν) be finite sets equipped with the discrete σ-algebra and measures μ and ν. Let $f : X \to Y$ be a measure-preserving function. Then there exists a disintegration $r : Y \rightsquigarrow X$ of μ over ν consistent with f unique ν-a.e.

In fact, a formula for the disintegration is

$$r_{xy} := \begin{cases} \mu_x \delta_{yf(x)} / \nu_y & \text{if } \nu_y > 0 \\ 1/|X| & \text{otherwise} \end{cases}$$

Completely positive maps and $*$-homomorphisms

Let $\mathcal{M}_n(\mathbb{C})$ denote the C^*-algebra of complex $n \times n$ matrices.

Definition

A *completely positive map* $\varphi : \mathcal{M}_n(\mathbb{C}) \rightsquigarrow \mathcal{M}_m(\mathbb{C})$ is a function for which there exists a finite set of linear transformations $\{V_i : \mathbb{C}^n \to \mathbb{C}^m\}$ such that

$$\varphi = \sum_{i=1}^{\text{finite}} \mathrm{Ad}_{V_i} \qquad \left(\varphi(A) = \sum_{i=1}^{\text{finite}} V_i A V_i^\dagger \right).$$

From finite sets to finite-dimensional C^*-algebras

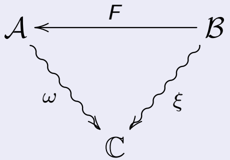
There is a (contravariant) functor from finite sets and transition kernels to finite-dimensional C^*-algebras and completely positive maps.

category theory	classical/ commutative	quantum/ noncommutative	physics/ interpretation
object	set	C^*-algebra	phase space observables
$\rightarrow$ morphism	function	$*$-homomorphism	deterministic process
$\rightsquigarrow$ morphism	transition kernel	completely positive map	non-deterministic process
monoidal product	cartesian product $\times$	tensor product $\otimes$	combining systems
$\rightsquigarrow$ to/from monoidal unit	probability measure	C^*-algebra state/ density matrix	physical state

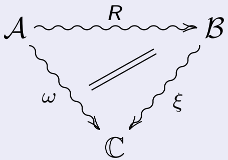
Non-commutative disintegrations

Definition

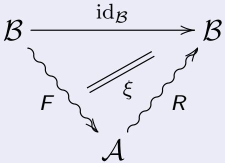
Let $(\mathcal{A}, \omega)$ and $(\mathcal{B}, \xi)$ be C^*-algebras equipped with states. Let $F : \mathcal{B} \to \mathcal{A}$ be a $*$-homomorphism such that the diagram on the right commutes.



A *disintegration* of ω over ξ consistent with F is a completely positive map $R : \mathcal{A} \rightsquigarrow \mathcal{A}$ such that



and



the latter diagram signifying commutativity ξ-a.e.

Existence and uniqueness of disintegrations

Existence is **not guaranteed** in the non-commutative setting.

Theorem

Fix $n, p \in \mathbb{N}$. Let

$$\begin{array}{ccc} \mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{\;F\;} & \mathcal{M}_n(\mathbb{C}) \\ & {\scriptstyle \operatorname{tr}(\rho\,\cdot\,)\equiv\omega} \searrow \quad \swarrow {\scriptstyle \xi\equiv\operatorname{tr}(\sigma\,\cdot\,)} & \\ & \mathbb{C} & \end{array}$$

be a commutative diagram with F the $*$-homomorphism given by the block diagonal inclusion $F(A) = \operatorname{diag}(A, \ldots, A)$. A disintegration of ω over ξ consistent with F exists **if and only if** *there exists a density matrix $\tau \in \mathcal{M}_p(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$.*

Example 1: Einstein-Rosen-Podolsky

Theorem

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \& \qquad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the diagonal map. Then* $\mathrm{tr}(\sigma A) = \mathrm{tr}(\rho F(A))$ *for all* A *but there does not exist a disintegration of* ρ *over* σ *consistent with* F*.*

Example 1: Einstein-Rosen-Podolsky

Theorem

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \& \qquad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the diagonal map. Then* $\mathrm{tr}(\sigma A) = \mathrm{tr}(\rho F(A))$ *for all* A *but there does not exist a disintegration of* ρ *over* σ *consistent with* F*.*

Proof.

ρ is entangled (not separable) and therefore cannot be expressed as the tensor product of any two 2×2 matrices. □

Example 2: Diagonal density matrices

Theorem

Fix $p_1, p_2, p_3, p_4 \geq 0$ *with* $p_1 + p_2 + p_3 + p_4 = 1$, $p_1 + p_3 > 0$, *and* $p_2 + p_4 > 0$. *Let*

$$\rho = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \qquad \& \qquad \sigma = \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix}$$

be density matrices and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the diagonal map. Then* $\mathrm{tr}(\sigma A) = \mathrm{tr}(\rho F(A))$ *for all* A. *Furthermore, there exists a disintegration of* ρ *over* σ *consistent with* F *if and only if*

$$p_1 p_4 = p_2 p_3.$$

Thank you

Thank you for your attention

The drawing of the sphere is due to Tomasz M. Trzeciak