Non-commutative disintegration

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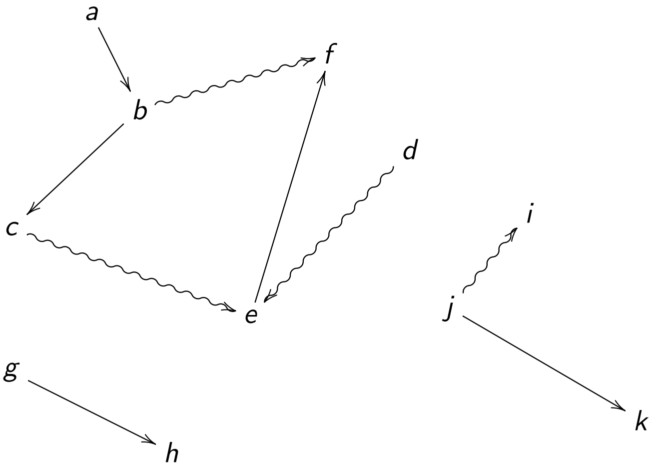
1. Deterministic and non-deterministic processes

2. Quantum disintegrations and regular conditions probabilities
   - Completely positive maps, $*$-homomorphisms, and states
   - Non-commutative disintegrations
   - Existence and uniqueness
   - Examples

3. Classical disintegrations
   - Classical disintegrations: formal definition
   - Classical disintegrations: intuition
   - Diagrammatic disintegrations
   - Classical disintegrations exist and are unique a.e.

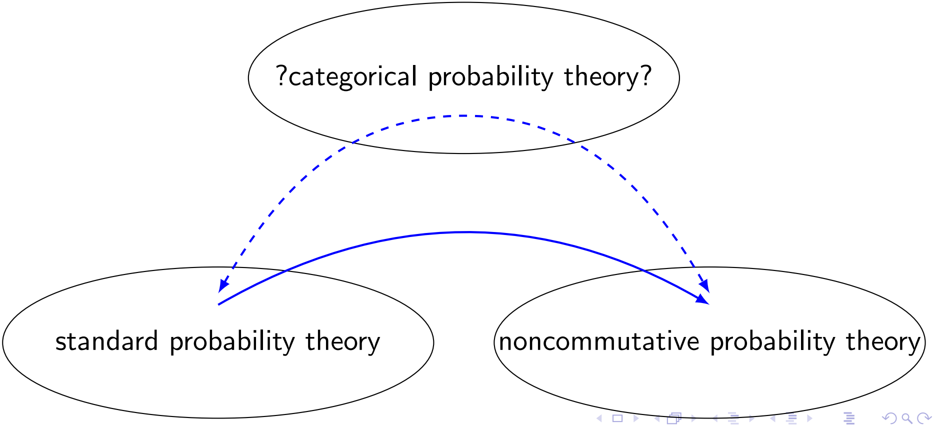
Category theory as a theory of processes

Objects are systems and arrows are processes. Processes can be deterministic $\to$ or non-deterministic $\rightsquigarrow$ and they can be composed.



Overview

Formulating concepts in probability theory categorically (diagrammatically) enables one to abstract these concepts to contexts beyond their initial domain. We will apply these ideas to non-commutative probability theory via finite-dimensional C^*-algebras (direct sums of matrix algebras).



Finite-dimensional C^*-algebras

- Let $\mathcal{M}_n(\mathbb{C})$ denote the set of complex $n \times n$ matrices. It is an example of a C^*-algebra: we can add and multiply $n \times n$ matrices, the operator norm gives a norm, and A^* is the conjugate transpose of A.

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- In particular, $\mathbb{C}^X$, functions from a finite set X to $\mathbb{C}$, is a *commutative* C^*-algebra. A basis for this algebra as a vector space is $\{e_x\}_{x \in X}$ defined by $e_x(x') := \delta_{xx'}$ (Kronecker delta function).

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- If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$ can be viewed as $n \times n$ matrices with entries in $\mathcal{A}$. It has a natural C^*-algebra structure.

Completely positive maps, $*$-homomorphisms, and states

Definition

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Too much?

If that was too much info, just remember this:

$\mathcal{A} \xleftarrow{F} \mathcal{B}$	$\mathcal{A} \overset{\varphi}{\rightsquigarrow} \mathcal{B}$	$\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C} \overset{\xi}{\leftsquigarrow} \mathcal{B}$
$*$-homomorphism	completely positive map	states
deterministic	stochastic	nc probability

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- A $*$-homomorphism $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ exists if and only if $m = np$ for some $p \in \mathbb{N}$. When this happens, there exists a unitary $m \times m$ matrix U (unitary means $UU^* = \mathbb{1}_m$) such that

$$F(A) = U \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix} U^* \text{ for all } A \in \mathcal{M}_n(\mathbb{C}).$$

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- (Choi's theorem) Let $R : \mathcal{M}_m(\mathbb{C}) \rightsquigarrow \mathcal{M}_n(\mathbb{C})$ be a completely positive map. Then there exist mn linear maps $R_i : \mathbb{C}^m \to \mathbb{C}^n$ such that $R = \sum_{i=1}^{mn} \mathrm{Ad}_{R_i}$, where $\mathrm{Ad}_{R_i}(A) := R_i A R_i^*$ for all $A \in \mathcal{M}_m(\mathbb{C})$.

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- If $\omega : \mathcal{M}_n(\mathbb{C}) \rightsquigarrow \mathbb{C}$ is a state, there exists a unique $n \times n$ positive matrix ρ such that $\mathrm{tr}(\rho) = 1$ and $\mathrm{tr}(\rho A) = \omega(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$. Such a matrix is called a *density matrix*.

More Examples (commutative algebras)

- Let X and Y be finite sets and let $F : \mathbb{C}^Y \to \mathbb{C}^X$ be a $*$-homomorphism. Then there exists a unique function $f : X \to Y$ such that $F(\varphi) = \varphi \circ f$ for all $\varphi \in \mathbb{C}^Y$.

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- Let X and Y be finite sets and let $R : \mathbb{C}^X \rightsquigarrow \mathbb{C}^Y$ be a completely positive map. Then there exists a unique *stochastic map/matrix* $r : Y \rightsquigarrow X$, i.e. a collection $\{r_{xy} := \langle e_y, R(e_x)\rangle \in \mathbb{R}\}_{\substack{x\in X\\ y\in Y}}$ such that

$$r_{xy} \geq 0, \quad \sum_{x\in X} r_{xy} = 1, \quad \& \quad R(\psi) = \sum_{y\in Y}\sum_{x\in X} \psi(x) r_{xy} e_y$$

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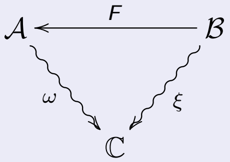
Non-commutative disintegrations

Definition (P-Russo)

Let $(\mathcal{A}, \omega)$ and $(\mathcal{B}, \xi)$ be C^*-algebras equipped with states. Let $F : \mathcal{B} \to \mathcal{A}$ be a $*$-homomorphism such that the diagram on the right commutes.



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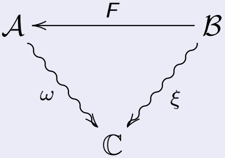

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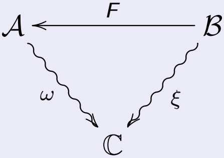

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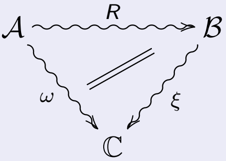
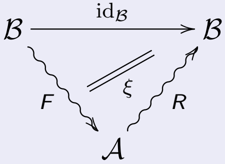


and



the latter diagram signifying commutativity ξ-a.e.

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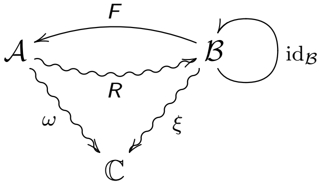
Very short answer: Category theory.
Slightly longer answer: It comes from a diagrammatic reformulation of a regular conditional probability. We will talk about this after our main theorem and two examples.



Existence and uniqueness of disintegrations

Theorem (P-Russo)

Fix $n, p \in \mathbb{N}$. Let

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\begin{array}{ccc}
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& {\scriptstyle \mathrm{tr}(\rho\,\cdot\,)\equiv\omega} \searrow \quad \swarrow {\scriptstyle \xi\equiv\mathrm{tr}(\sigma\,\cdot\,)} & \\
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be a commutative diagram with F the $*$-homomorphism given by the block diagonal inclusion $F(A) = \mathrm{diag}(A, \ldots, A)$.

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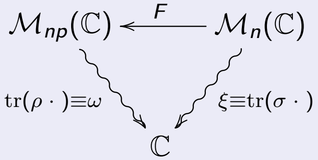

be a commutative diagram with F *the* $*$*-homomorphism given by the block diagonal inclusion* $F(A) = \mathrm{diag}(A, \ldots, A)$. *A disintegration of* ω *over* ξ *consistent with* F *exists* **if and only if** *there exists a density matrix* $\tau \in \mathcal{M}_p(\mathbb{C})$ *such that* $\rho = \tau \otimes \sigma$. *Furthermore, if a disintegration* R *exists, it is unique and satisfies* $R \circ F = \mathrm{id}_{\mathcal{M}_n(\mathbb{C})}$.

WGAD?

An important question:

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Example 1: Einstein-Podolsky-Rosen

Theorem (P-Russo)

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \& \qquad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the diagonal map.*

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Proof.

ρ is entangled (not separable) and therefore cannot be expressed as the tensor product of any two 2×2 density matrices. □

Example 2: Diagonal density matrices

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Fix $p_1, p_2, p_3, p_4 \geq 0$ *such that* $p_1 + p_2 + p_3 + p_4 = 1$. *Let*

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Theorem (P-Russo)

Fix $p_1, p_2, p_3, p_4 \geq 0$ *such that* $p_1 + p_2 + p_3 + p_4 = 1$. *Let*

$$\rho = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \qquad \& \qquad \sigma = \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix}$$

and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the block diagonal inclusion. Then* $\mathrm{tr}(\sigma A) = \mathrm{tr}(\rho F(A))$ *for all* A *and a disintegration of* ρ *over* σ *consistent with* F *exists if and only if* $p_1 p_4 = p_2 p_3$. *Furthermore, this disintegration is given by*

$$R = \mathrm{Ad}_{\sqrt{p_1+p_2}\left[\mathbb{1}_2 \quad 0\right]} + \mathrm{Ad}_{\sqrt{p_3+p_4}\left[0 \quad \mathbb{1}_2\right]}$$

Example 2: Diagonal density matrices

In more detail, if $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathcal{M}_4(\mathbb{C})$ with each $B_{ij} \in \mathcal{M}_2(\mathbb{C})$, then

$$\begin{aligned} R(B) &= (p_1 + p_2) \begin{bmatrix} \mathbb{1}_2 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \mathbb{1}_2 \\ 0 \end{bmatrix} \\ &\quad + (p_3 + p_4) \begin{bmatrix} 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbb{1}_2 \end{bmatrix} \\ &= (p_1 + p_2) B_{11} + (p_3 + p_4) B_{22}. \end{aligned}$$

Do I still have time?

If yes, proceed.
Else, conclude.

The formal definition of a disintegration

Definition

Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces and let $f : X \to Y$ be a measure-preserving map. A *regular conditional probability* is a transition kernel $r : Y \rightsquigarrow X$ for which there exists a ν-null set $N \in \Omega$ such that r_y is a probability measure for all $y \in Y \setminus N$ and

$$\mu\big(E \cap f^{-1}(F)\big) = \int_F r_y(E)\, d\nu(y) \qquad \forall\ E \in \Sigma \text{ and } \forall\ F \in \Omega.$$

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Here, $r : Y \rightsquigarrow X$ is a *transition kernel* means that r is a function $r : Y \times \Sigma \to [0, \infty]$ such that

i. $r(y, \cdot\,) : \Sigma \to [0, \infty]$ is a measure for all $y \in Y$ and
ii. $r(\,\cdot\,, E) : Y \to [0, \infty]$ is measurable for all $E \in \Sigma$.

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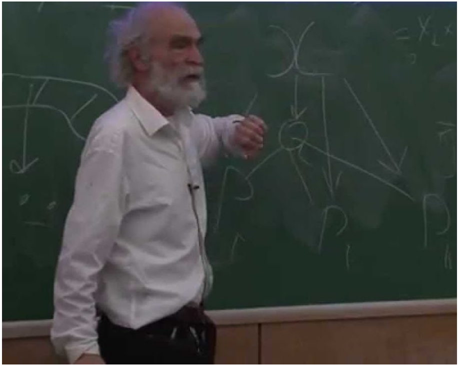
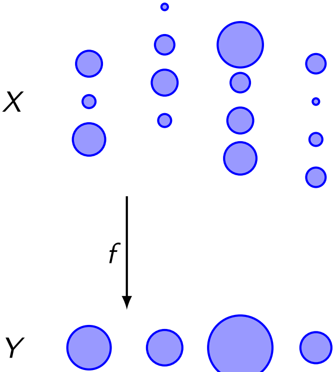
The notation $r_y(E) := r(y, E)$ is used.

Disintegration as a section

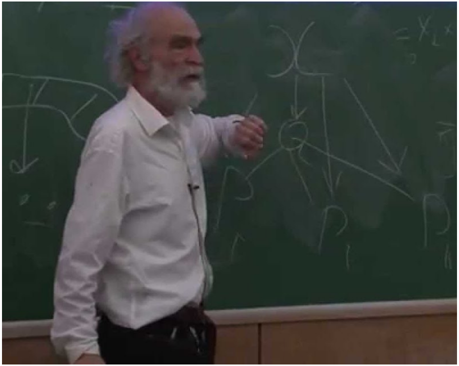
Let X and Y be finite sets equipped with probability measures. Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. f combines the water droplets and their volume (probabilities) add when they combine under f.

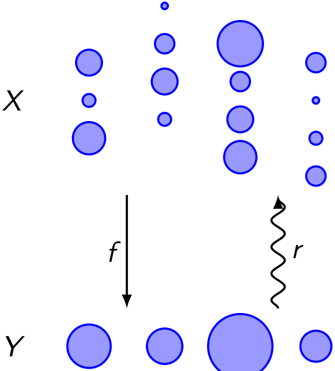




Disintegration as a section

Let X and Y be finite sets equipped with probability measures. Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. f combines the water droplets and their volume (probabilities) add when they combine under f. A disintegration $r : Y \rightsquigarrow X$ is a measure-preserving stochastic section of f.

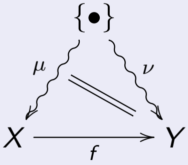




Disintegrations: diagrammatic definition

Definition

Let (X, μ) and (Y, ν) be finite probability spaces and let $f : X \to Y$ be a function such that the diagram on the right commutes.



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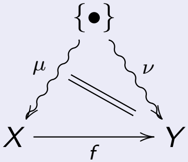
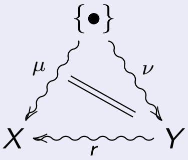
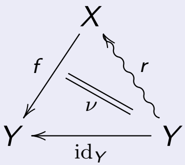


A *disintegration* (a.k.a. *regular conditional probability*) of μ over ν consistent with f is a stochastic map $r : Y \rightsquigarrow X$ such that



and



the latter diagram signifying commutativity ν-a.e.

Classical disintegrations exist and are unique a.e.

Theorem

Let (X, μ) and (Y, ν) be finite sets equipped with probability measures μ and ν. Let $f : X \to Y$ be a measure-preserving function. Then there exists a unique (ν-a.e.) disintegration $r : Y \rightsquigarrow X$ of μ over ν consistent with f.

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In fact, a formula for the disintegration is

$$r_{xy} := \begin{cases} \mu_x \delta_{yf(x)} / \nu_y & \text{if } \nu_y > 0 \\ 1/|X| & \text{otherwise} \end{cases}$$

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In summary, our theorem (almost!)* generalizes this existence and uniqueness result to the finite-dimensional non-commutative setting.

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In summary, our theorem (almost!)* generalizes this existence and uniqueness result to the finite-dimensional non-commutative setting.
*Almost because we are still finalizing the theorem for direct sums of matrix algebras.

Thank you!

Thank you for your attention!