5/7/2015 **Homotopy theory – A simple introduction to more advanced topics**

0) Topology, spaces, and homotopy

Def'n A topological space is a pair (X, τ) with X a set and τ a subset of the power set of X satisfying

i) $\emptyset \in \tau$ [Note: elements of τ are called open sets]

ii) $X \in \tau$

iii) $\bigcup_{\alpha \in A} \mathcal{U}_\alpha \in \tau$ for any collection $\{\mathcal{U}_\alpha \in \tau\}$

iv) $\bigcap_{i=1}^{n} \mathcal{U}_i \in \tau$ for any finite collection $\{\mathcal{U}_i \in \tau\}$

This defn is pretty abstract and not always easy to work with. We'll give a more geometric one in a minute but first we need basic examples.

ex (a) ($\mathbb{R}^n$, and all open subsets )

(b) $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ with "subspace topology"

(c) $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ "

(d) $\mathrm{int}(D^n) = \{x \in \mathbb{R}^n \mid |x| < 1\}$ "

Def'n A morphism f from (X, τ) to (Y, σ) is a function $f : X \to Y$ of sets s.t. $f^{-1}(\mathcal{U}) \in \tau$ for all $\mathcal{U} \in \sigma$. Morphisms are usually called continuous maps. If there exists a morphism $g : (Y, \sigma) \to (X, \tau)$ such that

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "id_X"'] & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, "g"] \arrow[rr, "id_Y"'] & & Y \end{tikzcd}$$

commute, then f is said to be an isomorphism, a.k.a. homeomorphism.

In this case (X,τ) and (Y,σ) are said to be <u>isomorphic</u> / <u>homeomorphic</u> and we write $(X,\tau) \cong (Y,\sigma)$.

<u>Def'n</u> An <u>n-cell</u> is a topological space homeomorphic to $\mathrm{int}(D^n)$. A <u>cell</u> is an n-cell for some n. 

<u>Fact</u> $\mathrm{int}(D^n) \cong \mathrm{int}(D^m)$ if and only if $n = m$.

n is the <u>dimension</u>.

<u>Def'n</u> A <u>cell decomposition</u> of a topological space (X,τ) is a family $\mathcal{E} = \{ e_\alpha \mid \alpha \in A \}$ of subspaces of X such that each e_α is a cell and $X = \coprod_{\alpha \in A} e_\alpha$ (the disjoint union of sets).

<u>Def'n*</u> A topological space (X,τ) is <u>Hausdorff</u> if for any two distinct points $x, x' \in X$ there exist open sets $\mathcal{U}, \mathcal{U}' \in \tau$ s.t. $x \in \mathcal{U}$, $x' \in \mathcal{U}'$ and $\mathcal{U} \cap \mathcal{U}' = \phi$.

<u>Def'n</u> A <u>CW complex</u> is a triple $(X,\tau,\mathcal{E})$ of Hausdorff topological space and a cell decomposition $\mathcal{E}$ of (X,τ) satisfying

i) for each n-cell e there is a map $\Phi_e : D^n \to X$ with $\mathrm{int}(D^n) \hookrightarrow D^n \xrightarrow{\Phi_e} X$ a homeomorphism ont and $S^{n-1} \hookrightarrow D^n \xrightarrow{\Phi_e} X$ has its image in cells over lower dimension than n

ii)* for any cell $e \in \mathcal{E}$, $\bar{e} := \bigcap \{ \mathcal{U}^c \equiv X \setminus \mathcal{U} \mid e \subset \mathcal{U}$ and $\mathcal{U} \in \tau \}$ intersects only finitely many cells.

iii)* A subset $\mathcal{U}$ is open if and only if $(\mathcal{U}^c \cap \bar{e})^c$ is open for each $e \in \mathcal{E}$.

<u>ex</u> anything you can draw by hand $\cong \mathbb{RP}^2$

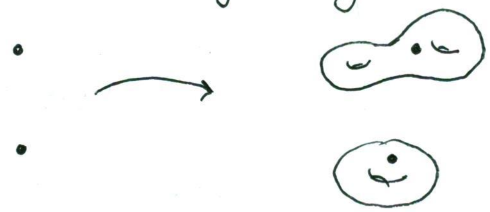
, , , , manifolds, orbifolds

1) Homotopy and homotopy groups

<u>Def'n</u> Let $(X,\tau) \overset{f}{\underset{g}{\rightrightarrows}} (Y,\sigma)$ be two maps. A <u>homotopy from f to g</u> is a map $H: X \times I \longrightarrow Y$ ($I \equiv \frac{1}{2}D^1 + \frac{1}{2}$) such that $H(x,0) = f(x)$ & $H(x,1) = g(x)$.

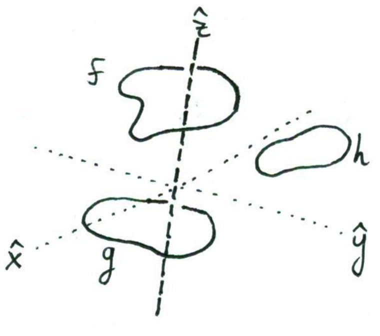
<u>ex</u> (a) take $X = D^0$ (a single point) and Y to be anything (such as S^2). Then a map from X to Y is just a point. A homotopy from one such map to another is a path from the first point to the other

A homotopy might not exist depending on Y

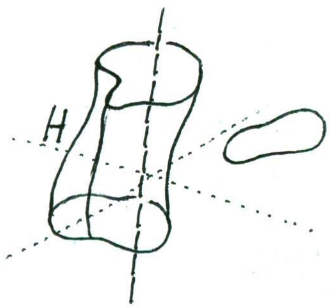


In this case, we say that Y is not path connected. If, however, for any two maps of points into Y there is a homotopy from one to the other then Y is said to be <u>path connected</u>.

(b) take $X = S^1$ and Y anything (such as $\mathbb{R}^3 \setminus$ line). Consider three maps ($\hat{z}$ axis is removed)



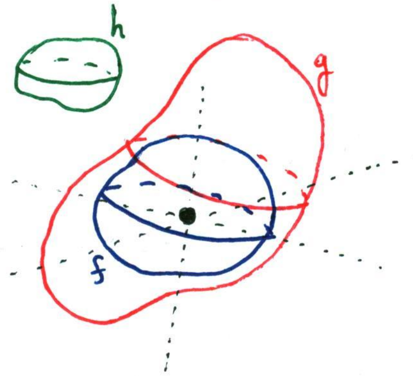
Then a homotopy from f to g can be drawn (and g to f) but not between f&h nor g & h



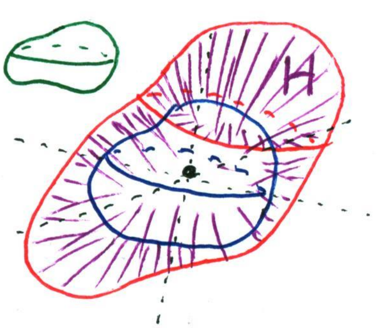
So we conclude that not every two circles are homotopic in $\mathbb{R}^3 \setminus$ line (However, it is path connected).

(c) take $X = S^2$ and Y anything (such as $\mathbb{R}^3 \setminus$ pt)

Consider three maps



Then we have a completely analogous situation as in example (b)



Fact Homotopy gives an equivalence relation on the set of maps from (X,τ) to (Y,σ). Namely $f \sim g$ if there exists a homotopy from f to g. We write the set of equivalence classes as $[X,Y]$.

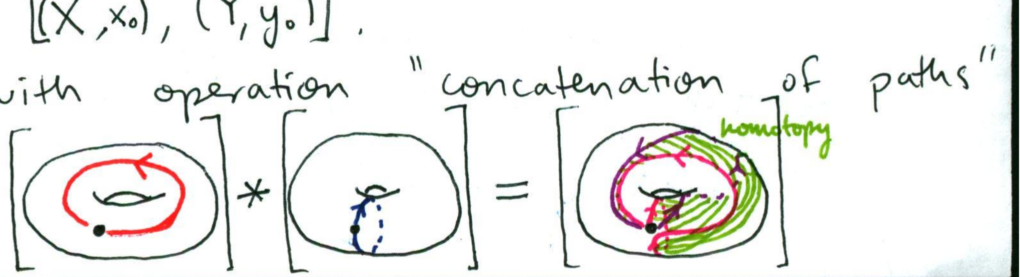
Def'n Let (X,τ) and (Y,σ) be topological spaces, $A \subset X$ (with the subspace topology), and $(X,\tau) \underset{g}{\overset{f}{\rightrightarrows}} (Y,\sigma)$ two maps. A <u>homotopy relative to A from f to g</u> is a homotopy from f to g that satisfies (in addition) $H(a,t) = f(a) = g(a)$ for all $t \in I$ (in particular f and g must be equal on A).

Rmk $A = \emptyset$ is the special case of ordinary homotopy. $A = pt$ will be used to define π_1.

Fact Homotopy relative to a subspace gives an equivalence relation on maps as above.

The definitions also make sense for pointed spaces (topological spaces with a basepoint — morphisms send basepoints to basepoints). In this case we write $[(X,x_0),(Y,y_0)]$.

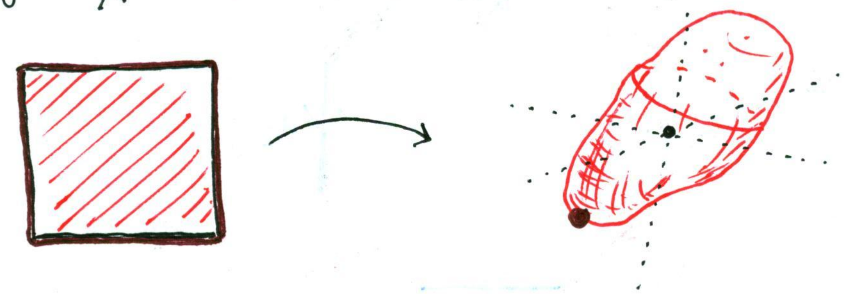
Thm $[(S^1,1),(X,x_0)]$ with operation "concatenation of paths" is a group, called the fundamental group of (X,x_0).



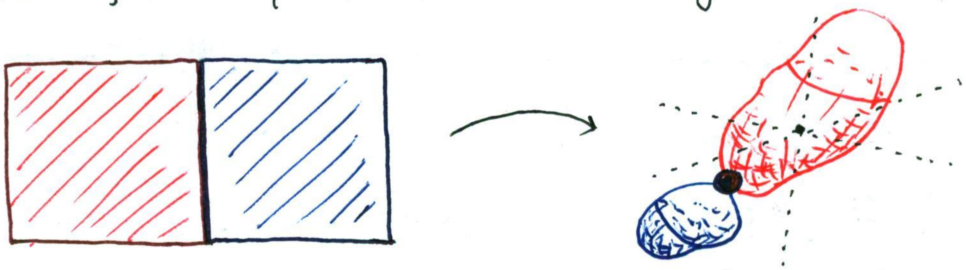
We sometimes write $[(S^1,1),(X,x_0)]$ as $\pi_1(X,x_0)$ or even $\pi_1(X)$ when x_0 is understood. Why the 1?

<u>Def'n/Notation</u> We write $\pi_n(X,x_0) := [(S^n,1),(X,x_0)]$.

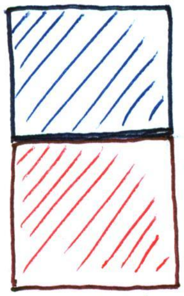
<u>Thm</u> $\pi_n(X,x_0)$ is an abelian group for all $n>1$. It is called the <u>n^{th} homotopy group</u> of (X,x_0). The concatenation is defined as follows. It is easier to visualize it if instead we view a map from S^n to X as a map $I^n \equiv I \times \cdots \times I$ to X that sends $\partial(I\times\cdots\times I)$ to x_0.



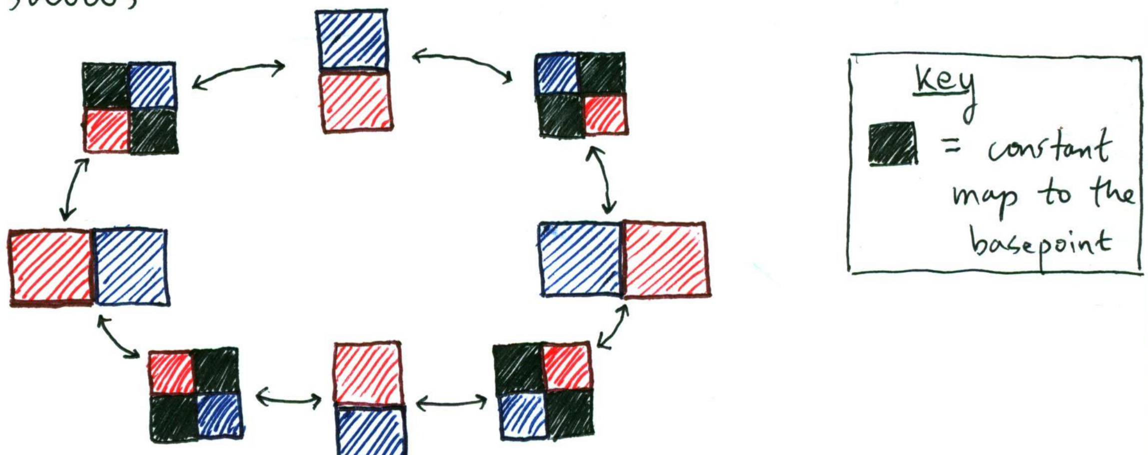
Then for the composition of two such spheres we just paste them together



Of course, we could have also done



These two choices are homotopic as the following cartoon shows



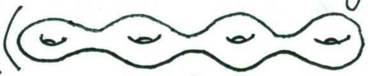
This also shows $\pi_2(X,x_0)$ is abelian (same works for π_n)

ex (a) $\pi_n((*,*)) \cong \{*\}$ for all n

(b) $\pi_1((S^1,1)) \cong \mathbb{Z}$ (winding number)

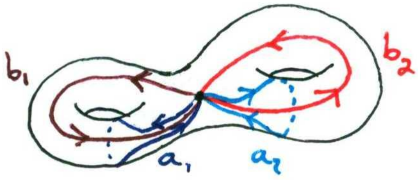
(c) $\pi_n((S^1,1)) \cong \{*\}$ for all $n > 1$

(d) $\pi_n((S^m,1)) \cong \begin{cases} \{*\} & \text{for } n < m \\ \mathbb{Z} & \text{for } n = m \\ \text{unknown in general} & \text{for } n > m \end{cases}$

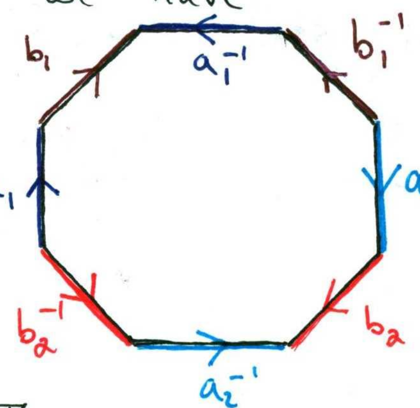
(f) $\pi_1(($  $, *)) \cong \left\langle a_1, \dots, a_g, b_1, \dots, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] \right\rangle$

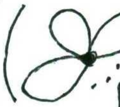
genus g

since for ex $g = 2$ we have



$\cong$



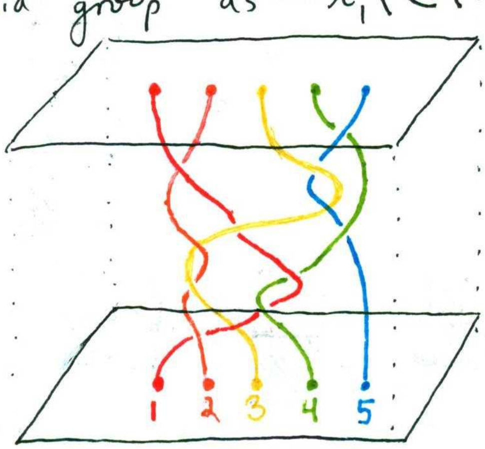
(e) $\pi_1(($  $, \bullet)) \cong \mathbb{Z} * \cdots * \mathbb{Z}$ the free group on p generators. If we call the generators $a_1, \dots, a_p$ we may also write $\langle a_1, \dots, a_p \rangle$

(g) $\pi_2((G,e)) \cong \{*\}$ & $\pi_3((G,e)) \cong \mathbb{Z}$ for a semi-simple (or some nice property) Lie group. Also, $\pi_1((G,e))$ is abelian for <u>any</u> Lie group.

(h) $\pi_3((S^2,1)) \cong \mathbb{Z}$

(i) Pure braid group as $\pi_1(C(n;\mathbb{R}^2), (1,2,\dots,n))$ where

A loop in $C(n;\mathbb{R}^2)$ is a pure braid.



$C(n;\mathbb{R}^2)$ is the configuration space of n distinguishable particles in the plane. Specifically, $C(n;\mathbb{R}^2) = \{(x_1, \dots, x_n) \in \mathbb{R}^2 \times \dots \times \mathbb{R}^2 \mid x_i \neq x_j \text{ for all } i \neq j\}$.

2) Fibrations and bundles

<u>Def'n</u> A surjective map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a <u>Serre fibration</u> if it satisfies the following property:

to any CW complex $(Z, \rho, \mathcal{E})$ with a map $g : (Z, \rho) \longrightarrow (X, \tau)$ and $h : Z \times I \longrightarrow Y$ fitting into the diagram

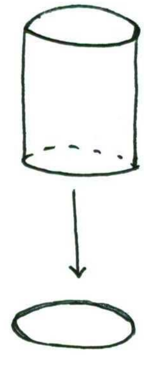
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, hook, "i_0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "h"'] & Y \end{tikzcd}$$

then there exists a map $\tilde{h} : Z \times I \longrightarrow X$ (called a lift of h) so that

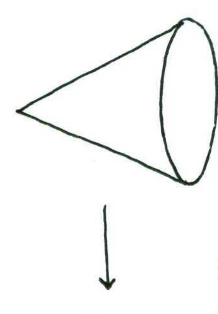
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, hook, "i_0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd}$$

commutes.

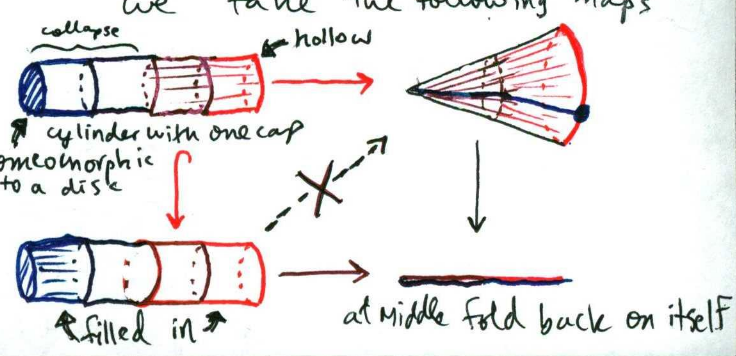
<u>ex</u> (a) Product fibrations $X \times I \longrightarrow X$



(b) <u>non</u> example



It's a nonexample because if we take $Z = D^2$ and we take the following maps

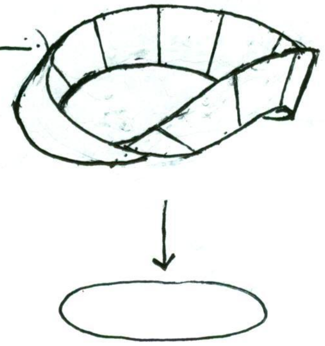


(c) <u>fiber bundles</u> over B with fiber F: $E \xrightarrow{\pi} B$

for every $b \in B$ there exists a $\mathcal{U}$ open in B with $b \in \mathcal{U}$ and a homeomorphism ψ_b fitting into

$$\begin{tikzcd} E\big|_{\mathcal{U}} \equiv \pi^{-1}(\mathcal{U}) \arrow[rr, "\psi_b"] \arrow[dr] & & \mathcal{U} \times F \arrow[dl] \\ & \mathcal{U} & \end{tikzcd}$$

examples (—)



Here $F = I$ & $B = S^1$

Möbius bundle

(=) $S^1 \longrightarrow S^3$
$\quad\quad\quad\downarrow$
$\quad\quad\quad\mathbb{C}P^1$

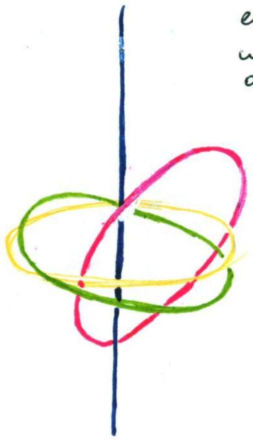
Dirac monopole bundle aka Hopf fibration aka Tautological bundle over $\mathbb{C}P^1 \cong S^2$

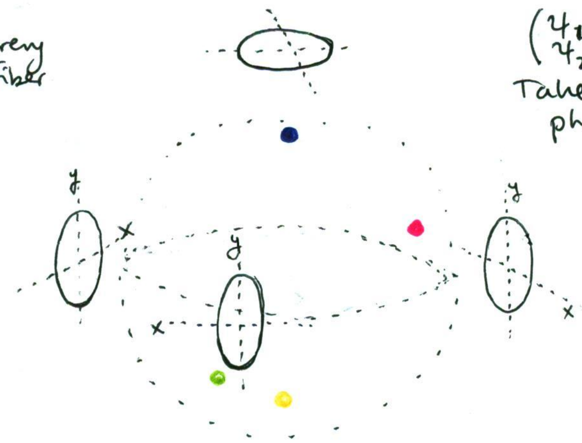
-Take $\mathbb{C}^2$ (nonrelativistic spin $\frac{1}{2}$)

$\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ state $|\psi_1|^2 + |\psi_2|^2 = 1$

Take ray - the phase is the S^1!

Some fibers ↓ each fiber is linked exactly with every other fiber



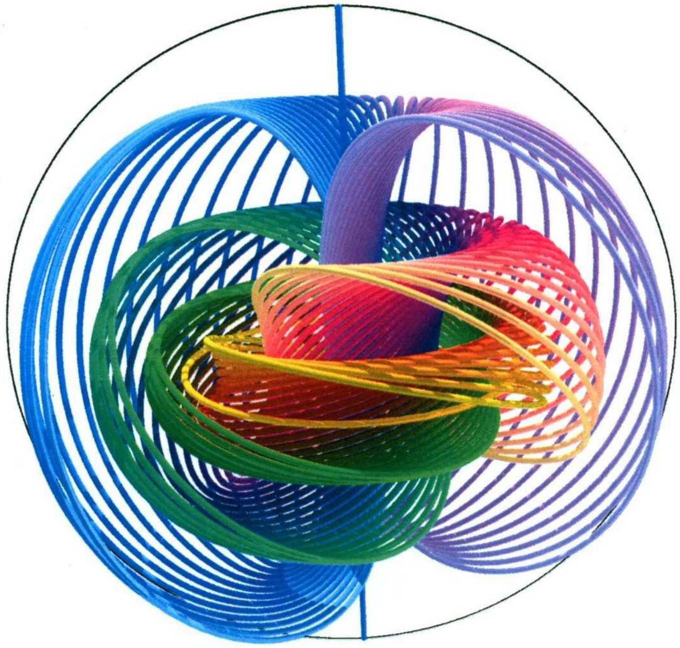


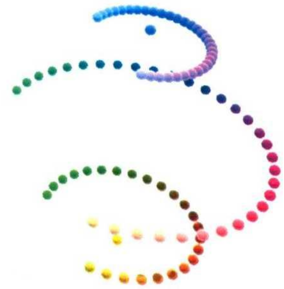
This can also be described as the lines in the tangent bundle "nematic" (since $\mathbb{R}P^1 \cong S^1$, it's a circle bundle)

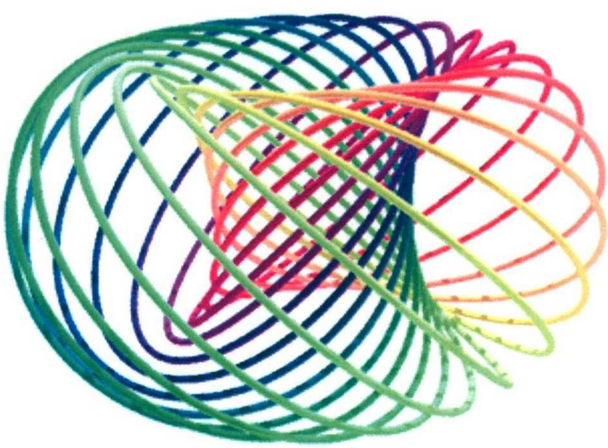
Fun fact: if we view this as a complex line bundle, tensoring with itself gives the tangent bundle of the (complex curve) manifold S^2. In other words, this is the square root of the tangent bundle.

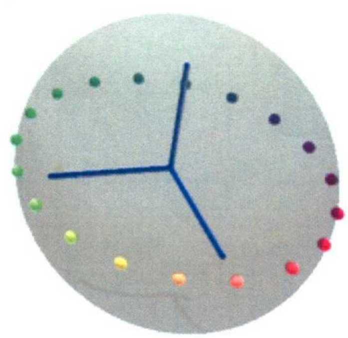
(d) Serre's based loop path fibration over a based space (X, x_0)

Let E = all paths starting at x_0. Define $E \xrightarrow{t} X$ to be the target map. Then $E \xrightarrow{t} X$ is a fibration with fiber $F = \Omega_{x_0} X$ the based loop space at x_0. Notice that E is contractible! We will use this fact later.









Images by Niles Johnson http://nilesjohnson.net/hopf.html

(e) <u>Covering spaces</u>

Continuous surjective map $E \xrightarrow{p} B$ s.t. to each $x \in B$ there exists an a neighborhood $\mathcal{U}$ with $x \in \mathcal{U}$ and $p^{-1}(\mathcal{U})$ a union of disjoint open sets in E with each mapping homeomorphically to $\mathcal{U}$.

<u>ex</u> The <u>universal cover</u> $\tilde{B} \to B$

<u>Def'n</u> For any covering map $E \to B$ there exists a unique covering map $\tilde{B} \to E$ s.t.

$$\begin{tikzcd} & \tilde{B} \arrow[dl] \arrow[d] \\ E \arrow[r] & B \end{tikzcd}$$

commutes

<u>Construction</u> $\tilde{B} = P_b B / \sim$ homotopy rel endpoints

Here $P_b B$ is the space of paths starting at the basepoint $b \in B$.

There are two important facts about Serre fibrations that we'll discuss soon, the second of which is a tool for computing π_n.

<u>Def'n</u> Let $X \xrightarrow{f} Y$ be a continuous map of topological spaces (or CW complexes). f is a <u>weak homotopy equivalence</u> if for every choice of $x_0 \in X$ the induced maps on homotopy groups $f_{*n} : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$, which are given on representatives by precomposition

,

are all isomorphisms. <u>Caution</u>: this does <u>not</u> give an equivalence relation.

<u>Theorem</u> Let $E \xrightarrow{p} B$ be a fibration. Then for any two points $b, b' \in B$ in the same path component of B, the fibers $p^{-1}(b)$ and $p^{-1}(b')$ are weakly homotopy equivalent.

<u>pf sketch</u> The fiber $p^{-1}(b)$ above $b \in B$ is part of a pullback diagram

$$\begin{tikzcd} p^{-1}(b) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \{b\} \arrow[r, hook] & B \end{tikzcd}$$

Define the <u>homotopy fiber over b</u>, denoted by F_b, to be the homotopy pullback of the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \{b\} \arrow[r, hook] & B \end{tikzcd}$$

which is defined to be the pullback of the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ P_b B \arrow[r, "t", "\text{target}"'] & B \end{tikzcd}$$

, which is just

$$F_b := \left\{ (e, \gamma) \in E \times P_b B \;\middle|\; t(\gamma) \equiv \gamma(1) = p(e) \right\}.$$

Then we have, by the universal property of F_b,

$$\begin{tikzcd} p^{-1}(b) \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[d] & & \\ \{b\} \arrow[dr, bend right] & F_b \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ & P_b B \arrow[r, "t"] & B \end{tikzcd}$$

a unique map $p^{-1}(b) \longrightarrow F_b$. The claim is that this map is a weak homotopy equivalence. Once this is proved (it's not difficult to prove that) we have for any two fibers $p^{-1}(b) \longrightarrow F_b$ & $p^{-1}(b') \longrightarrow F_{b'}$ but F_b and $F_{b'}$ are homotopy equivalent (not just weakly) since $P_b B \simeq P_{b'} B$. Thus we have a zigzag of weak homotopy equivalences $p^{-1}(b) \longrightarrow F_b \rightleftarrows F_{b'} \longleftarrow p^{-1}(b')$ showing that $p^{-1}(b)$ & $p^{-1}(b')$ are weakly homotopy equivalent. ■

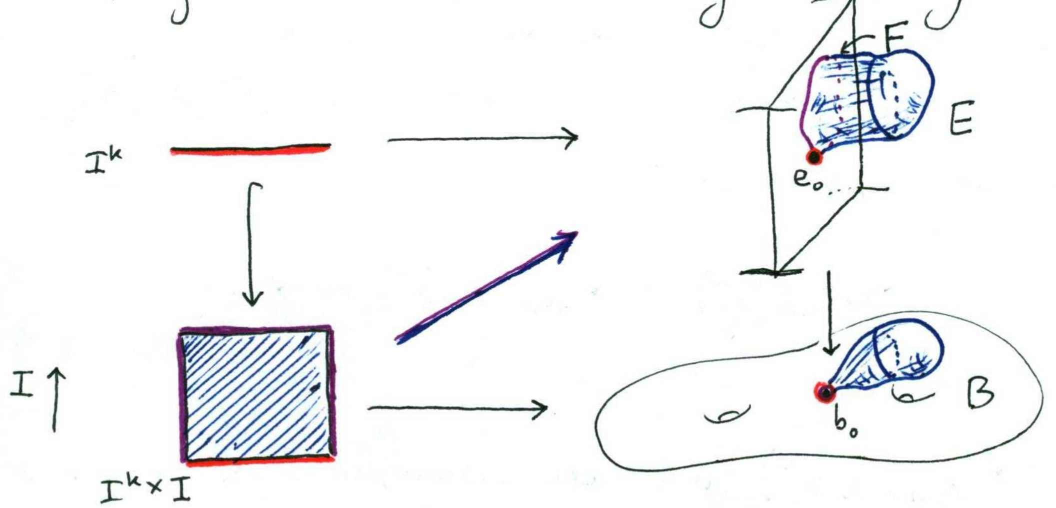
<u>Rmk</u> In particular, the fibers all have isomorphic homotopy groups.

<u>Theorem</u> Let $E \xrightarrow{p} B$ be a Serre fibration and let $e_0 \in E$ and $b_0 \in B$ be basepoints with $p(e_0) = b_0$. Let $E_{b_0} \equiv F$ denote the fiber over b_0. Let $i : E_{b_0} \longrightarrow E$ be the inclusion. Then, there exists a long exact sequence (l.e.s.)

$$\cdots \xrightarrow{p_{*n+1}} \pi_{n+1}(B) \xrightarrow{\delta_n} \pi_n(F) \xrightarrow{i_{*n}} \pi_n(E) \xrightarrow{p_{*n}} \pi_n(B) \xrightarrow{\delta_{n-1}} \pi_{n-1}(F) \longrightarrow \cdots$$

, which terminates when the statement no longer makes sense. The maps $\{\delta_n\}_n$ are called <u>connecting homomorphisms</u>.

They are defined by choosing a lift



(Note that $\partial(I^k \times I)$ gets sent to b_0)

and restricting to the boundary. One should check this is well-defined (using the definition of a Serre fibration). ∎

<u>Applications/examples</u>

(a) $\Omega_{x_0} X \longrightarrow P_{x_0} X$, $P_{x_0} X \longrightarrow X$. Note that $P_{x_0} X$ is contractible (the spaghetti map) so the l.e.s. says

$$\cdots \longrightarrow \underbrace{\pi_{n+1}(P_{x_0} X)}_{=0} \longrightarrow \pi_{n+1}(X) \longrightarrow \pi_n(\Omega_{x_0} X) \longrightarrow \underbrace{\pi_n(P_{x_0} X)}_{=0} \longrightarrow \cdots$$

$\Rightarrow$ $\pi_{n+1}(X) \longrightarrow \pi_n(\Omega_{x_0} X)$ are all isomorphisms.

(b) $V \longrightarrow E$, $E \longrightarrow X$ vector bundles. In this case, V is contractible so the l.e.s. says

$$\cdots \longrightarrow \underbrace{\pi_n(V)}_{=0} \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \underbrace{\pi_{n-1}(V)}_{=0} \longrightarrow \cdots$$

$\Rightarrow$ $\pi_n(E) \longrightarrow \pi_n(X)$ are all isomorphisms. Therefore, E and X are weakly homotopy equivalent.

(c) $S^1 \longrightarrow S^3$ Hopf fibration. Here is a much more interesting and nontrivial fibration.

$\qquad\qquad\downarrow$

$\qquad\quad S^2$ In this case, we know $\pi_n(S^1) = \mathbb{Z}\cdot\delta_{1n}$ for $n \geq 1$. Therefore, for $n \geq 2$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_{n+1}(S^1) & \longrightarrow & \pi_{n+1}(S^3) & \longrightarrow & \pi_{n+1}(S^2) & \longrightarrow & \pi_n(S^1) & \longrightarrow \cdots \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

$\Rightarrow \quad \pi_{n+1}(S^3) \longrightarrow \pi_{n+1}(S^2)$ are all isomorphisms for $n \geq 2$.

In particular, this gives us a proof that $\pi_3(S^2) \cong \mathbb{Z}$.

(d) $\pi_1(X, x_0) \longrightarrow \tilde{X}$ the universal cover. Here, the fiber

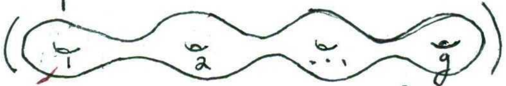
$\qquad\qquad\qquad\quad\downarrow$ is discrete so it has no nontrivial

$\qquad\qquad\qquad\quad X$ homotopy groups (let's suppose X is connected). By definition $\pi_1(\tilde{X}) = 0$.

Therefore, the l.e.s. says for all $n \geq 2$

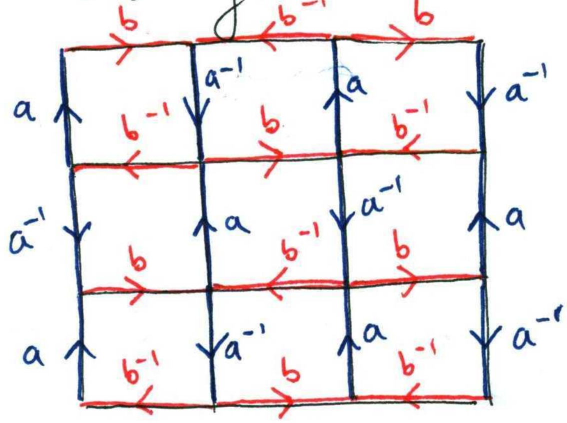
$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_n(\pi_1(X, x_0)) & \longrightarrow & \pi_n(\tilde{X}) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_{n-1}(\pi_1(X, x_0)) & \longrightarrow \cdots \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

$\Rightarrow \quad \pi_n(\tilde{X}) \longrightarrow \pi_n(X)$ are all isomorphisms for $n \geq 2$.

This has immediate interesting consequences (that might not be apriori obvious). For instance, it says that $\pi_n\Big($  $\Big) = 0$ for all $n \geq 2$.

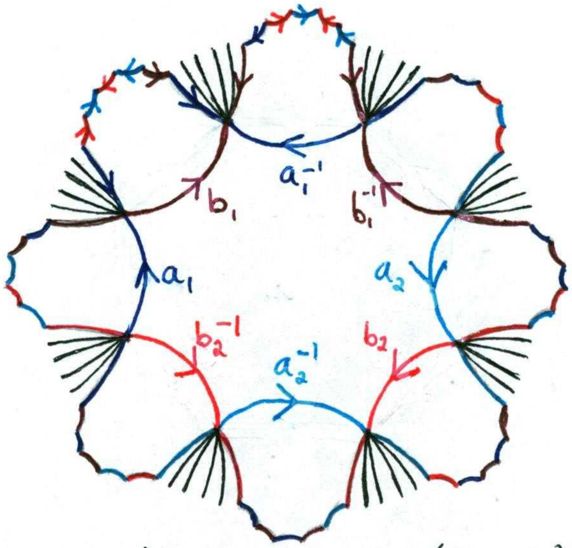
The reason is because $\mathbb{R}^2$ is the universal cover.

To do this any justice, let's just visually show why for the cases $g=1$ and $g=2$. For $g=1$, it's obvious



Each square maps onto the torus

For $g=2$, we need to use the hyperbolic disc to visualize why the plane covers the genus 2 surface. Just like in the $g=1$ case we reflect across each face and continously tile the disc in this way.



(e)
$$\begin{array}{ccc} C(n+1;\mathbb{R}^2\setminus\{(0,0)\}) & \longrightarrow & C(n;\mathbb{R}^2) \\ & & \Big\downarrow p_1 \\ & & \mathbb{R}^2 \end{array}$$

configuration space fibration. Here p_1 chooses the position of the first particle.

This example of a fibration was proved much more generally by Fadell & Neuwirth in 1962. Since $\mathbb{R}^2$ is contractible, it says

$$\cdots \longrightarrow \pi_m(C(n+1;\mathbb{R}^2\setminus\{0\})) \longrightarrow \pi_m(C(n;\mathbb{R}^2)) \longrightarrow \cdots$$

are isomorphisms for all m.

(f)
$$\begin{array}{ccc} \mathcal{U}(n-1) & \hookrightarrow & \mathcal{U}(n) \\ & & \Big\downarrow \\ & & S^{2n-1} \end{array}$$

defined by $A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and $B \longmapsto B\begin{pmatrix} 1+i\cdot 0 \\ 0+i\cdot 0 \\ \vdots \\ 0+i\cdot 0 \end{pmatrix}$ where we view S^{2n-1} as the unit sphere in $\mathbb{C}^n$.