The contraction mapping theorem

Fractals from iterations

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1. The notion of distance
   - Euclidean distance
   - Metric spaces
   - Towards the Hausdorff metric
   - The Hausdorff metric
2. Constructing fractals via iteration
   - Operations on subsets of $\mathbb{R}^2$
   - Sierpinski triangle
   - The contraction mapping theorem
3. Other applications of the contraction mapping theorem
   - Differential equations
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   - Differential equations

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$$\mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{d} [0, \infty)$$

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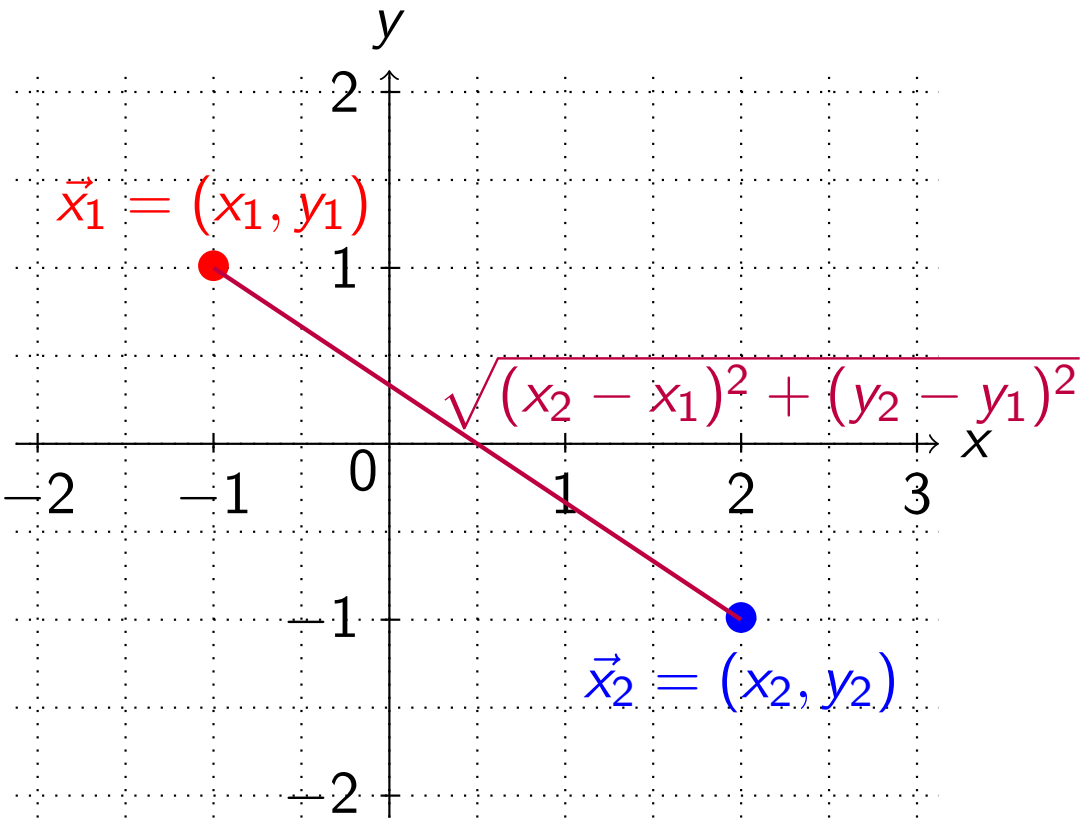

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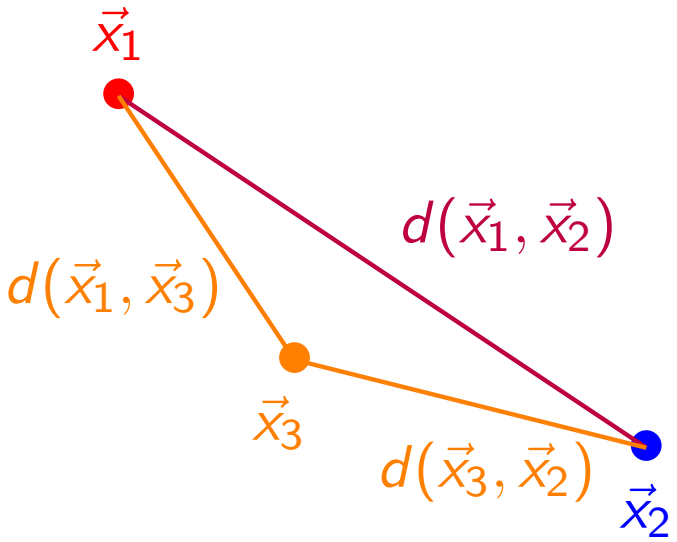

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Functions on an interval

An example is the set of continuous functions $C([a,b])$ on a closed interval $[a,b]$ using the distance

$$C([a,b]) \times C([a,b]) \to [0,\infty)$$
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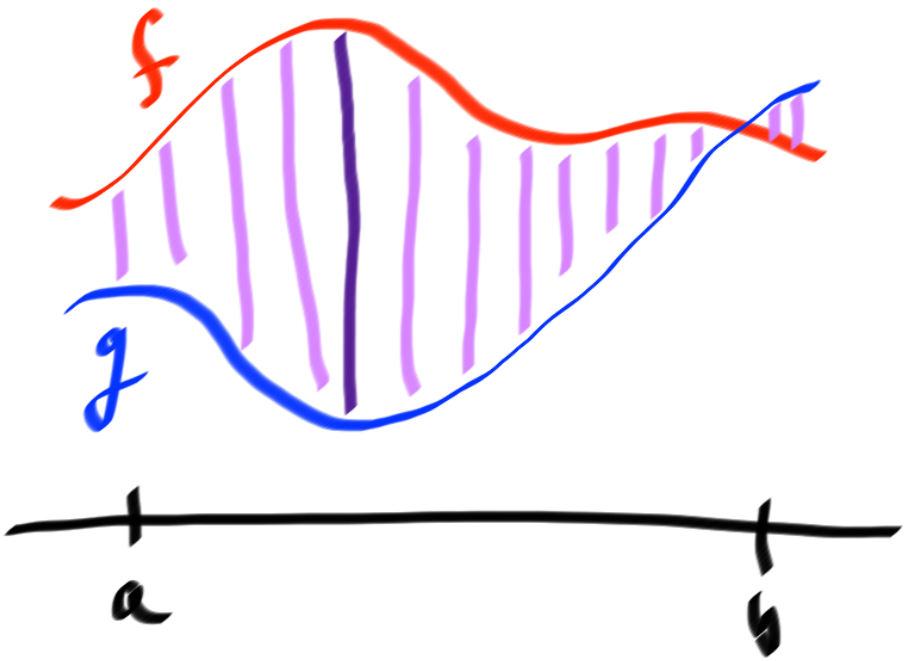

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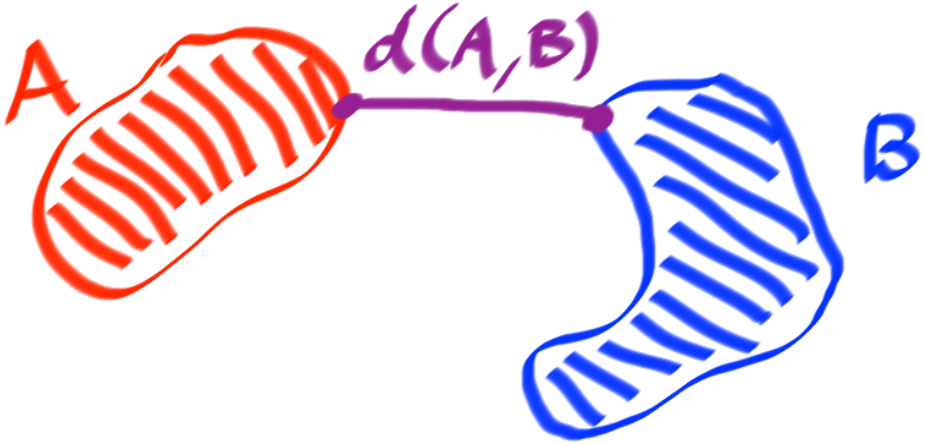

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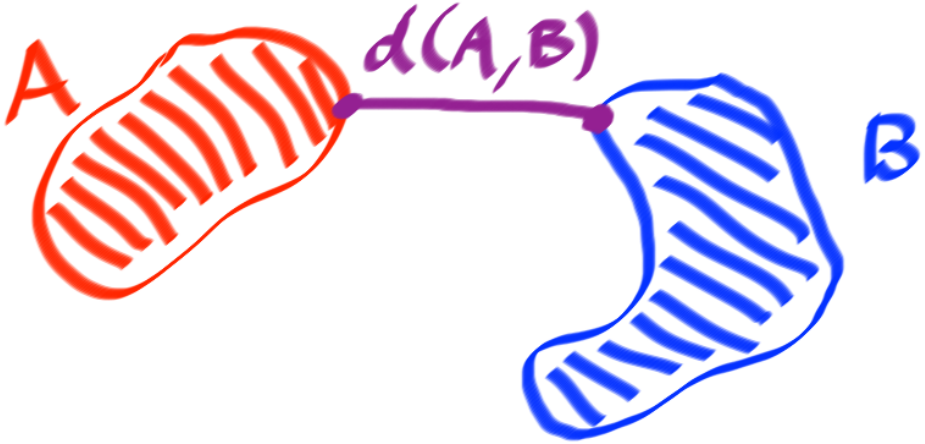

Does this give us a metric on $\mathcal{P}(\mathbb{R}^2)$?

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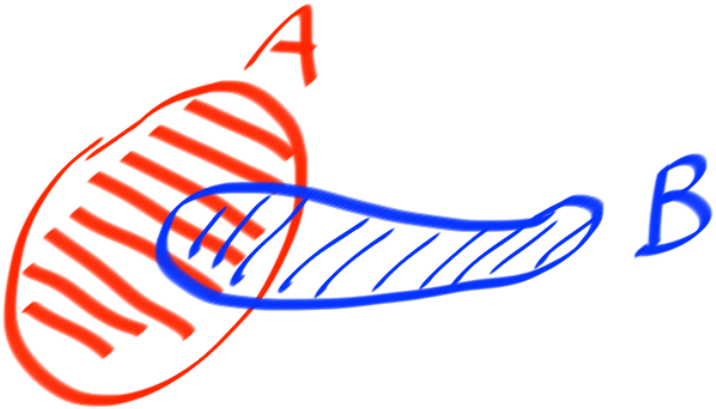

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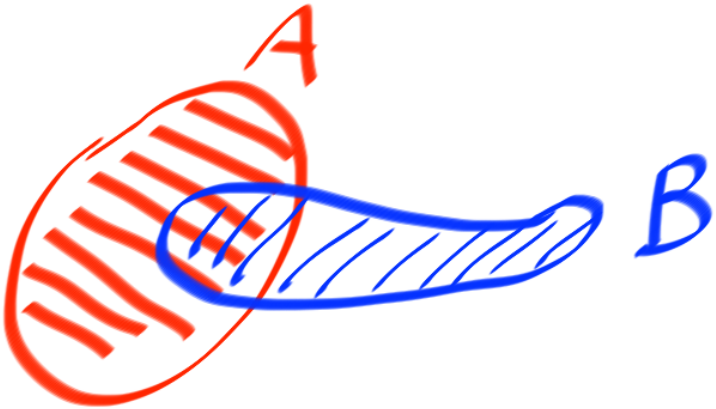

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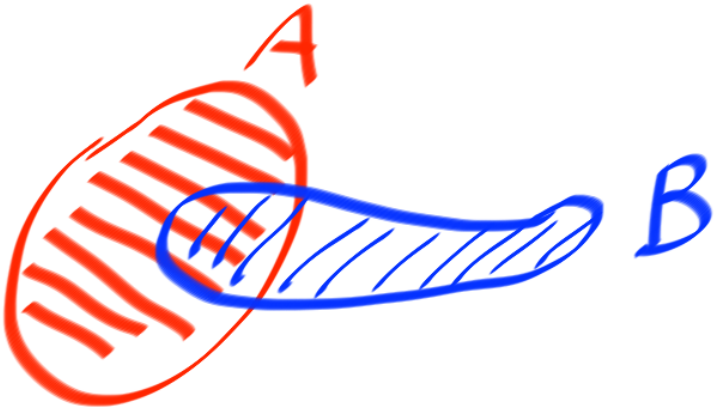

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$d(A, B) > 0$, $d(A, C) = 0$, and $d(C, B) = 0$ so $d(A, B) \leq d(A, C) + d(C, B)$ fails.

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(a) Barnsley Fern

(b) Real fern

The ϵ-collar

Definition

Let $A \subseteq \mathbb{R}^2$ and let $\epsilon > 0$. The <u>ϵ-collar</u> of A is the set

$$A_\epsilon := \left\{ x \in \mathbb{R}^2 \ : \ d(x, A) \leq \epsilon \right\}$$

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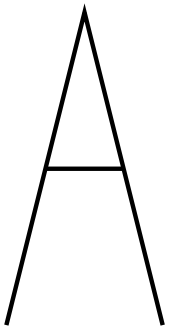

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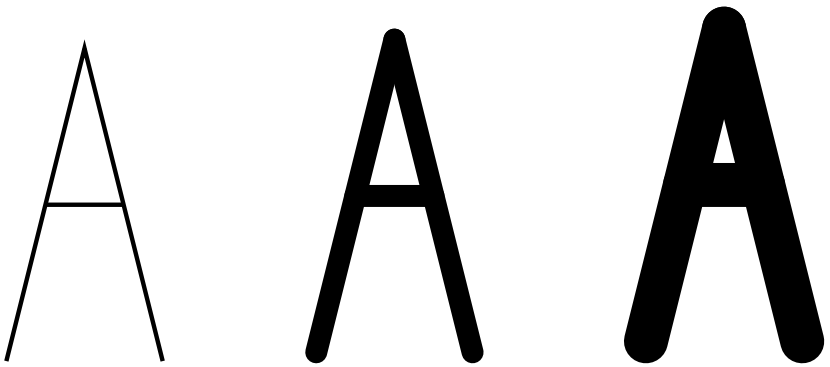

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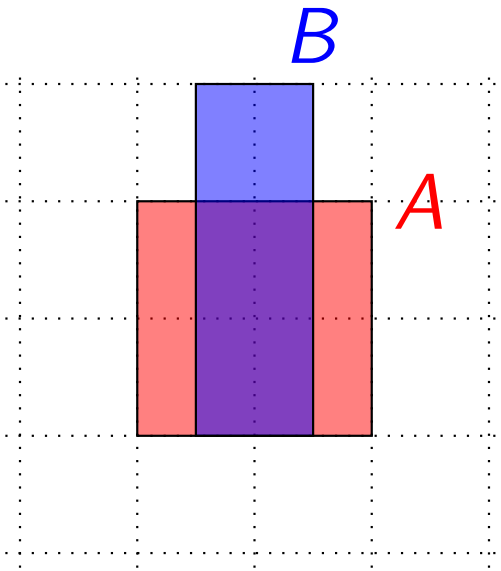
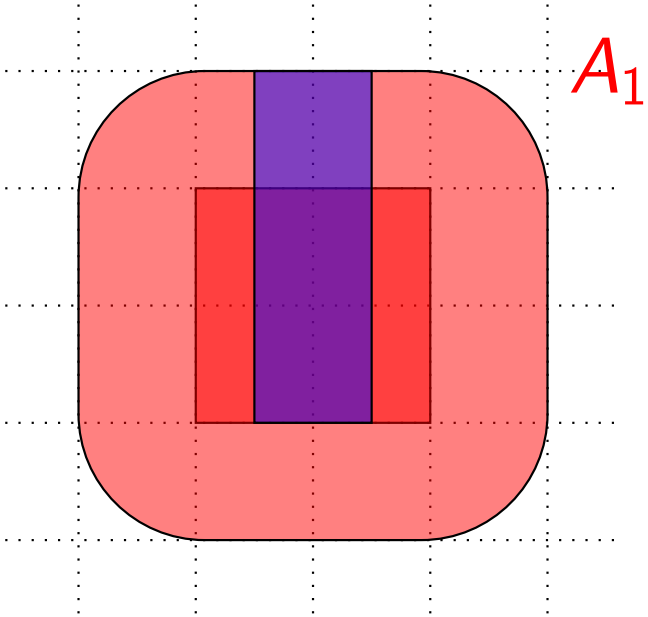
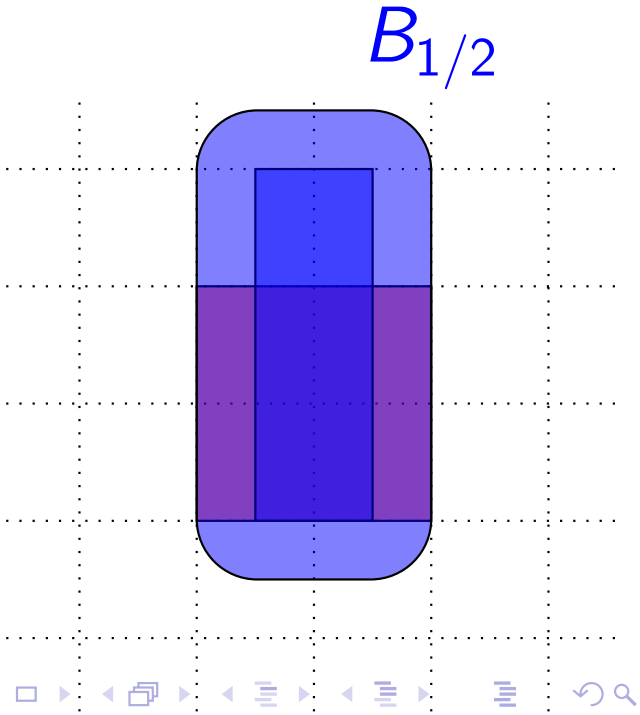
This is called the <u>*Hausdorff distance*</u>.

Example of the Hausdorff distance





$$d(A, B) = \frac{1}{2}$$



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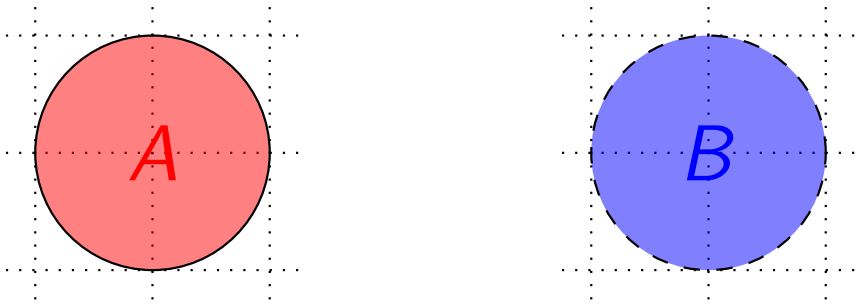

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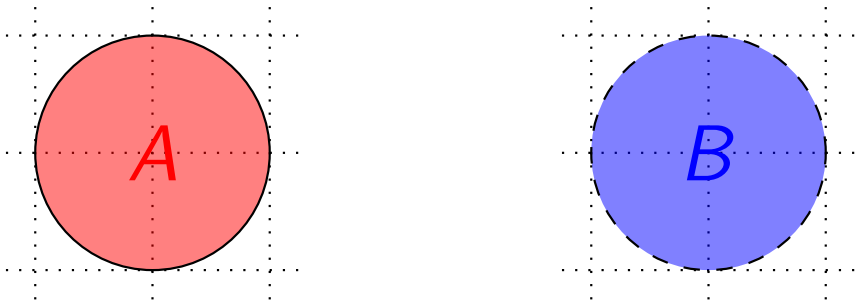

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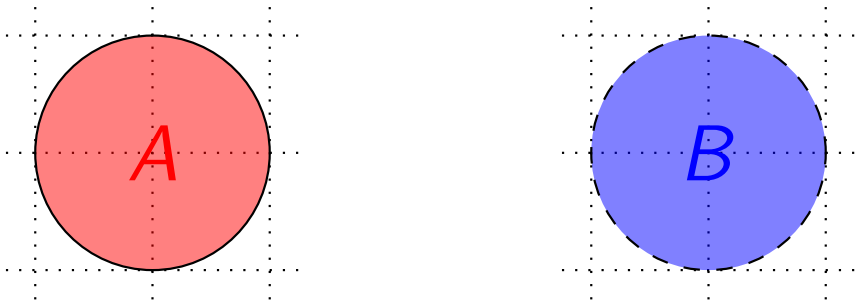

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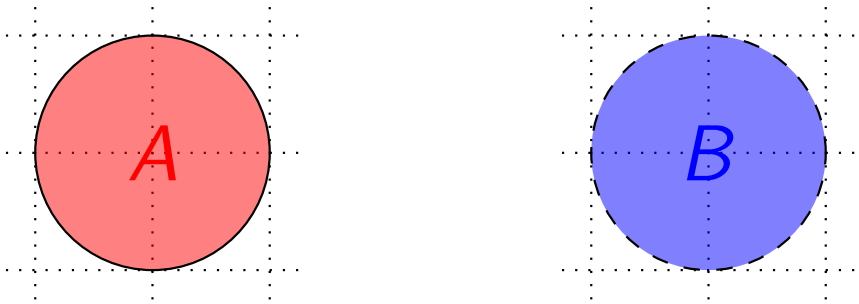

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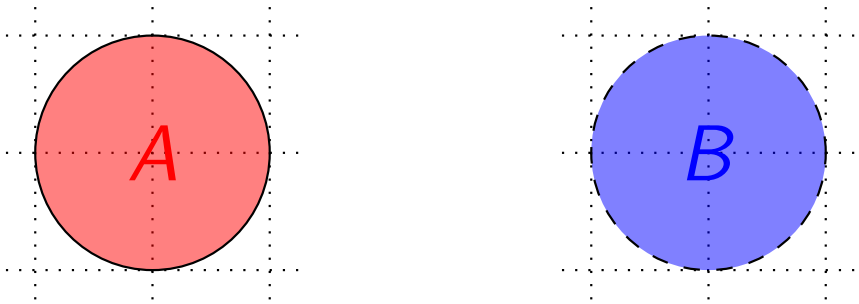

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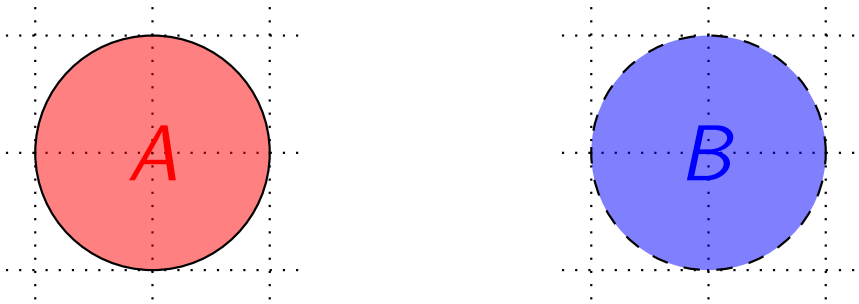

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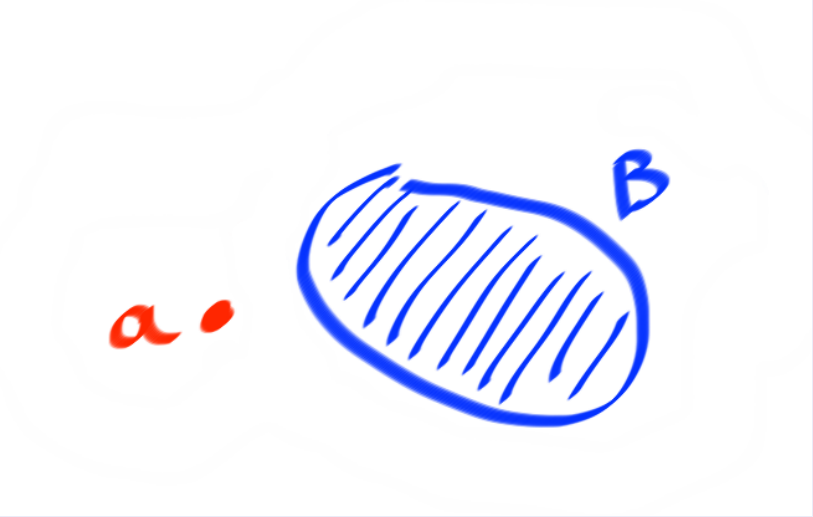

$\square$

The Hausdorff metric on compact subsets of $\mathbb{R}^2$

Theorem

The Hausdorff distance is a metric on the set X of all closed and bounded subsets of $\mathbb{R}^2$.

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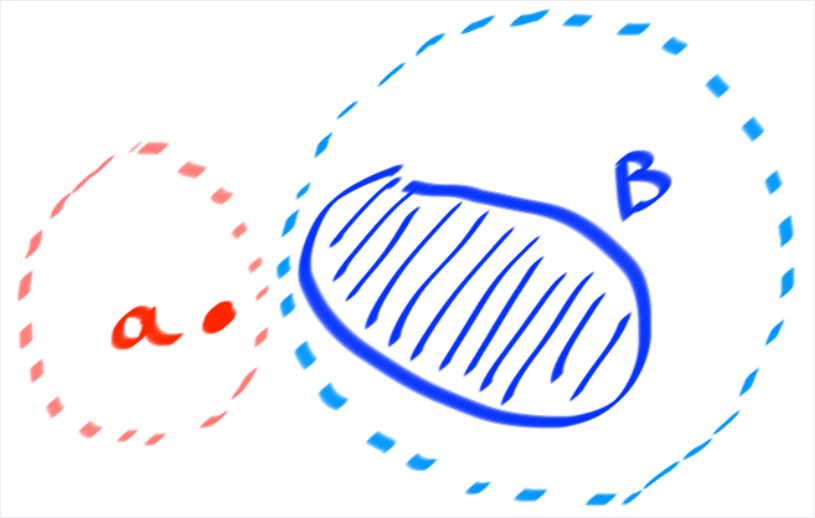

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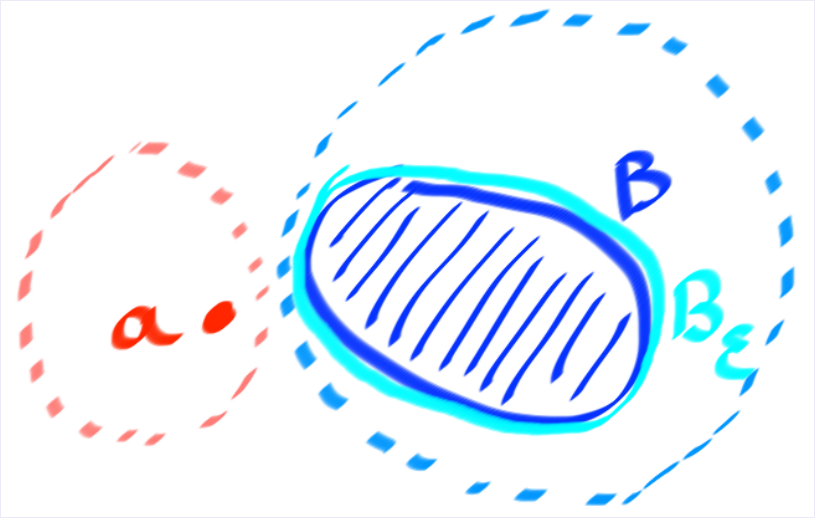

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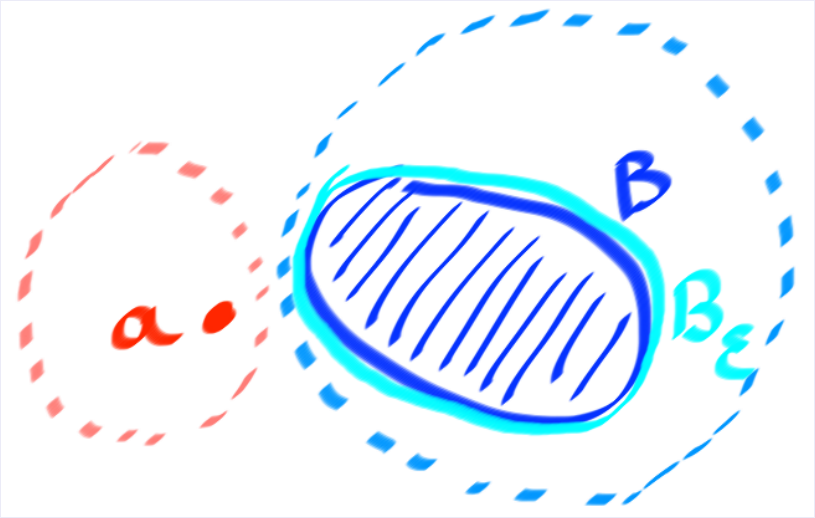

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Proof.

By a similar argument and by symmetry of d, one concludes $B \subseteq A$. Hence $B = A$. Of course that's not the end of the proof. You still have to show d satisfies the triangle inequality! Since this is a bit more technical, I'll let you think about it! $\square$

Operations on subsets of $\mathbb{R}^2$

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$$\mathbb{R}^2 \xrightarrow{f_1} \mathbb{R}^2$$
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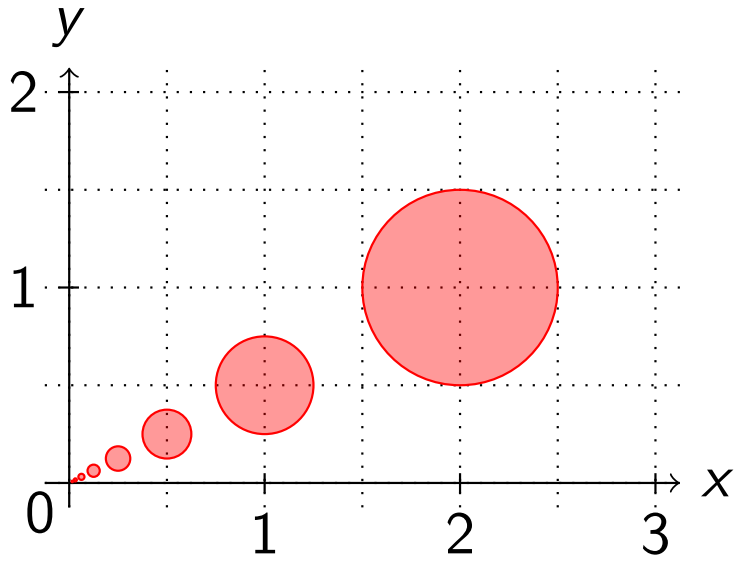

In the figure on the left, $A, F_1(A), F_1^2(A), \ldots, F_1^7(A)$ are all shown. Here, A is the disc of radius $\frac{1}{2}$ centered at the point $(2, 1)$.

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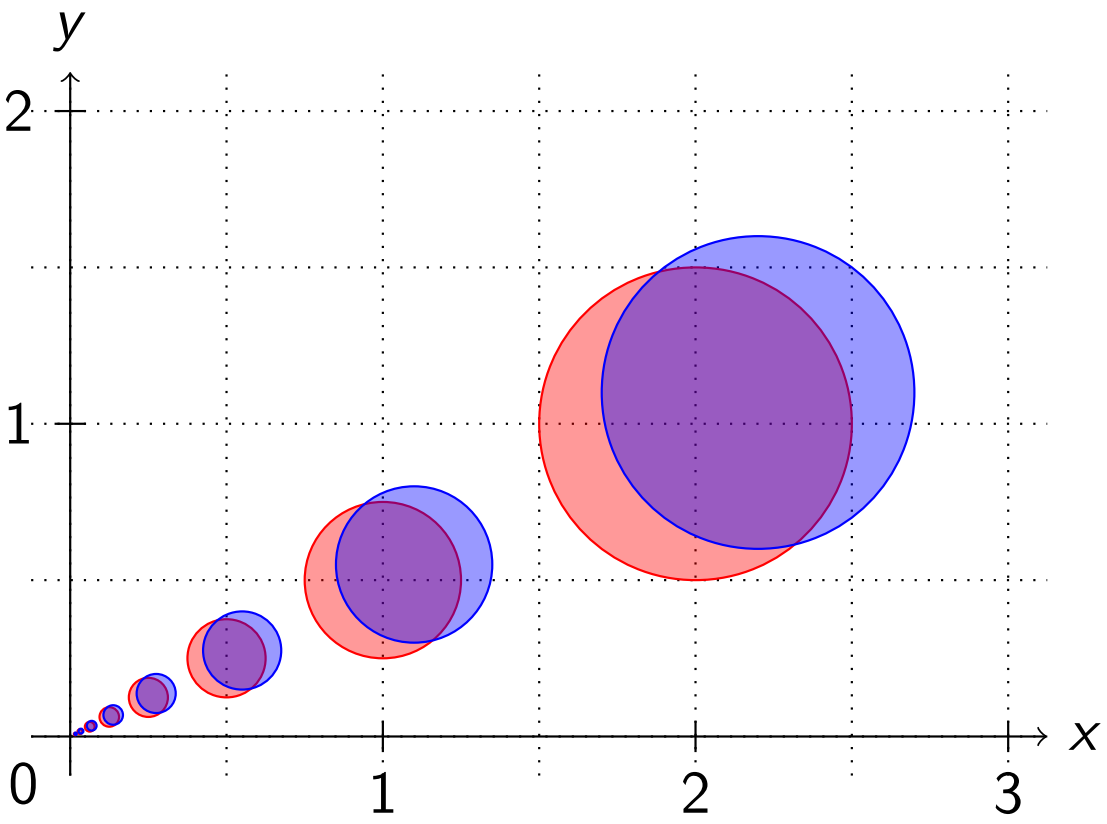

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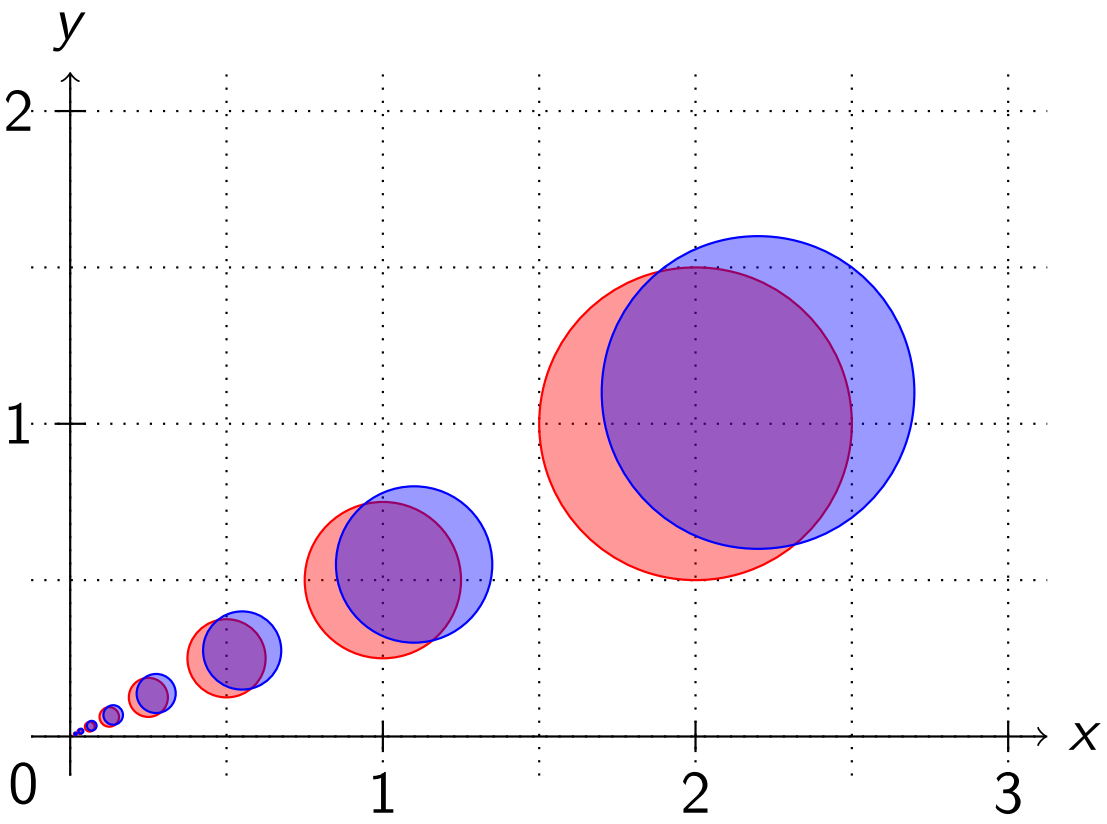


In fact, $d\Big(F_1(A), F_1(B)\Big) \leq \frac{1}{2} d(A, B)$.

Contractions on metric spaces

Definition

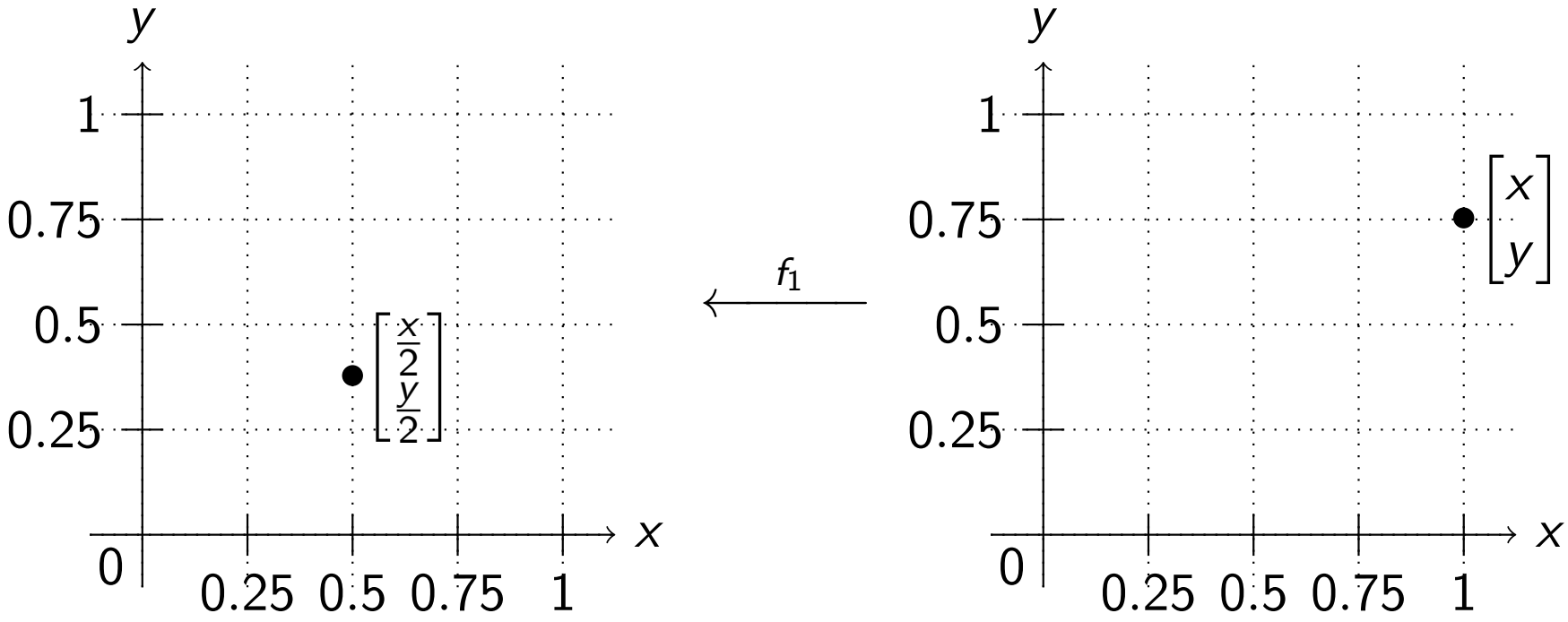
Let (X, d) be a metric space. A function $X \xrightarrow{F} X$ is a <u>*contraction*</u> *iff there exists a constant $c \in [0, 1)$ such that*

$$d\Big(f(x_1), f(x_2)\Big) \leq c\ d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

F_1

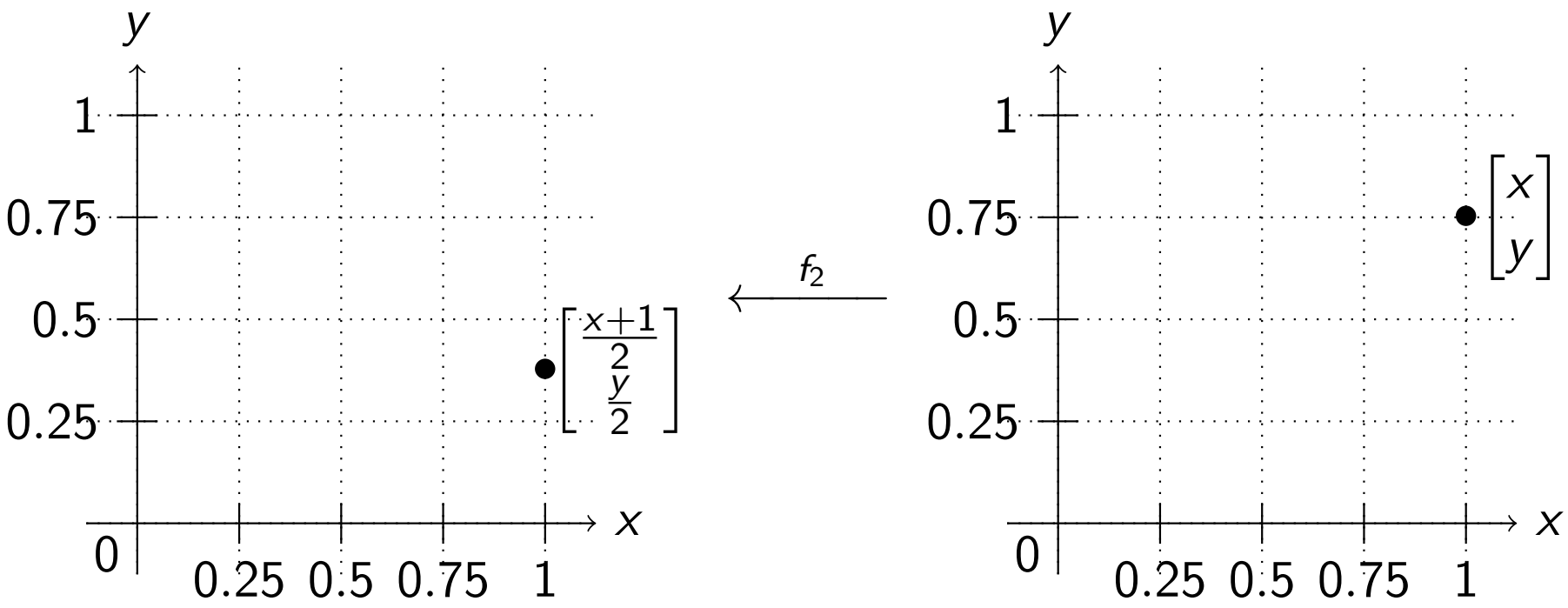
Let $\mathbb{R}^2 \xleftarrow{f_1} \mathbb{R}^2$ be the affine transformation defined earlier by



and let F_1 be the associated function on $\mathcal{H}$, closed and bounded subsets of $\mathbb{R}^2$.

F_2

Let $\mathbb{R}^2 \xleftarrow{f_2} \mathbb{R}^2$ be the affine transformation defined by



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F_3

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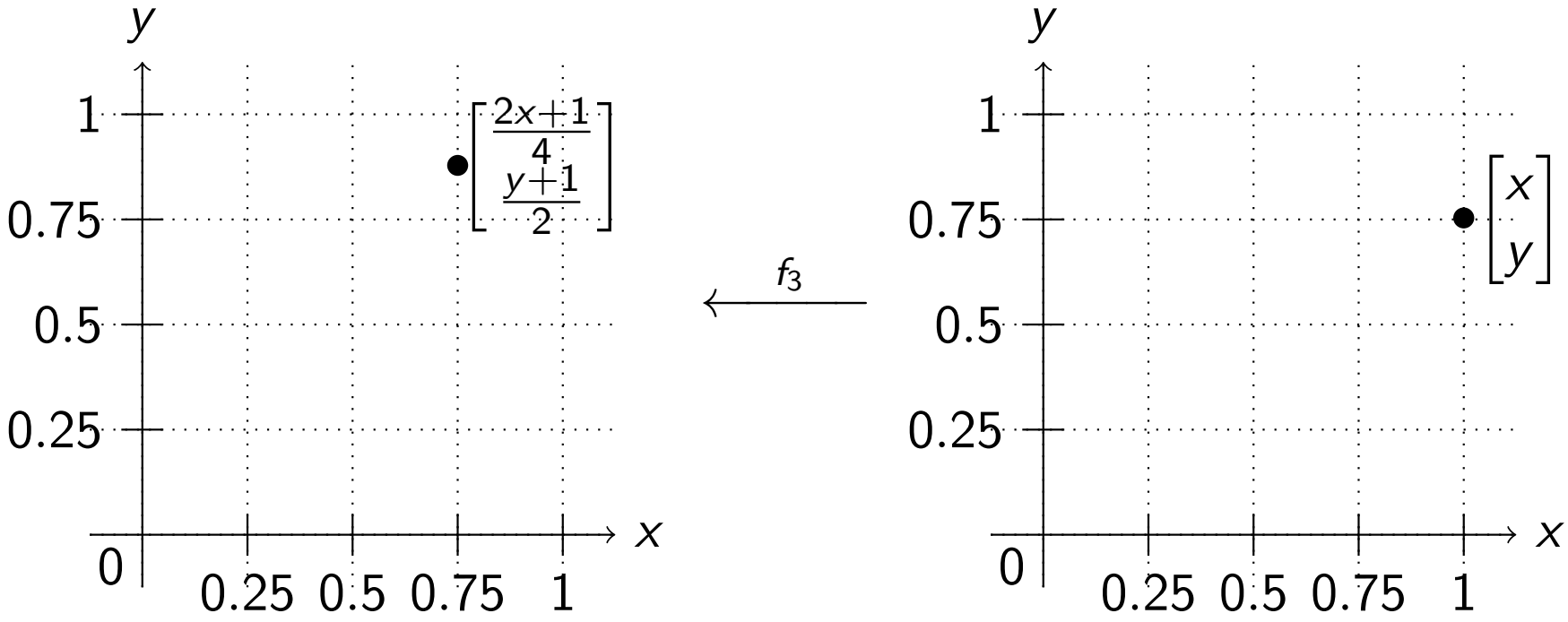

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Combining the three operations

Given any closed and bounded subset $S \subseteq \mathbb{R}^2$, we can perform the previous three operations as many times as we want on S to construct a new subset of $\mathbb{R}^2$.

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We can also take the union of the resulting subsets

$$\mathfrak{F}^n(S) := \bigcup_{i_1,\ldots,i_n=1}^{3} F_{i_n}(F_{i_{n-1}}(\cdots(F_{i_1}(S))\cdots))$$

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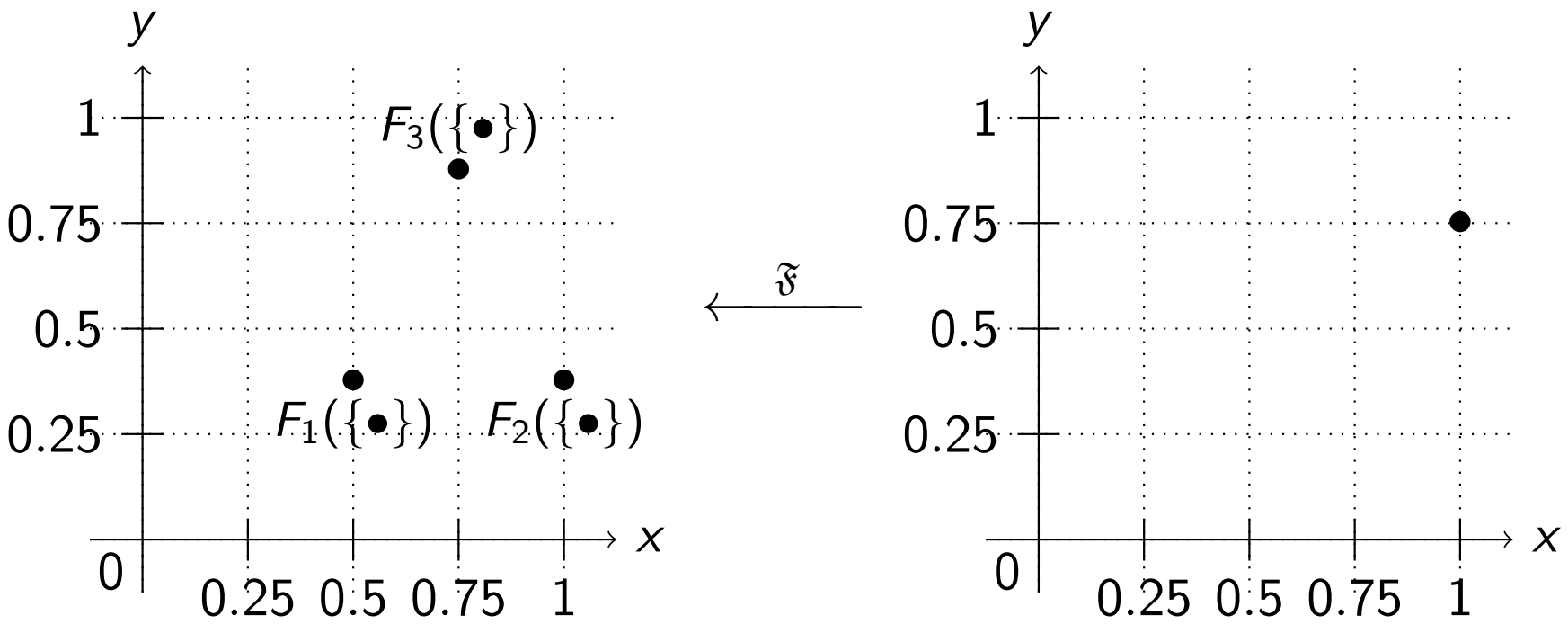
to generate a new subset that is also closed and bounded. Therefore, $\mathfrak{F} : \mathcal{H} \to \mathcal{H}$ defines a function on closed and bounded subsets.

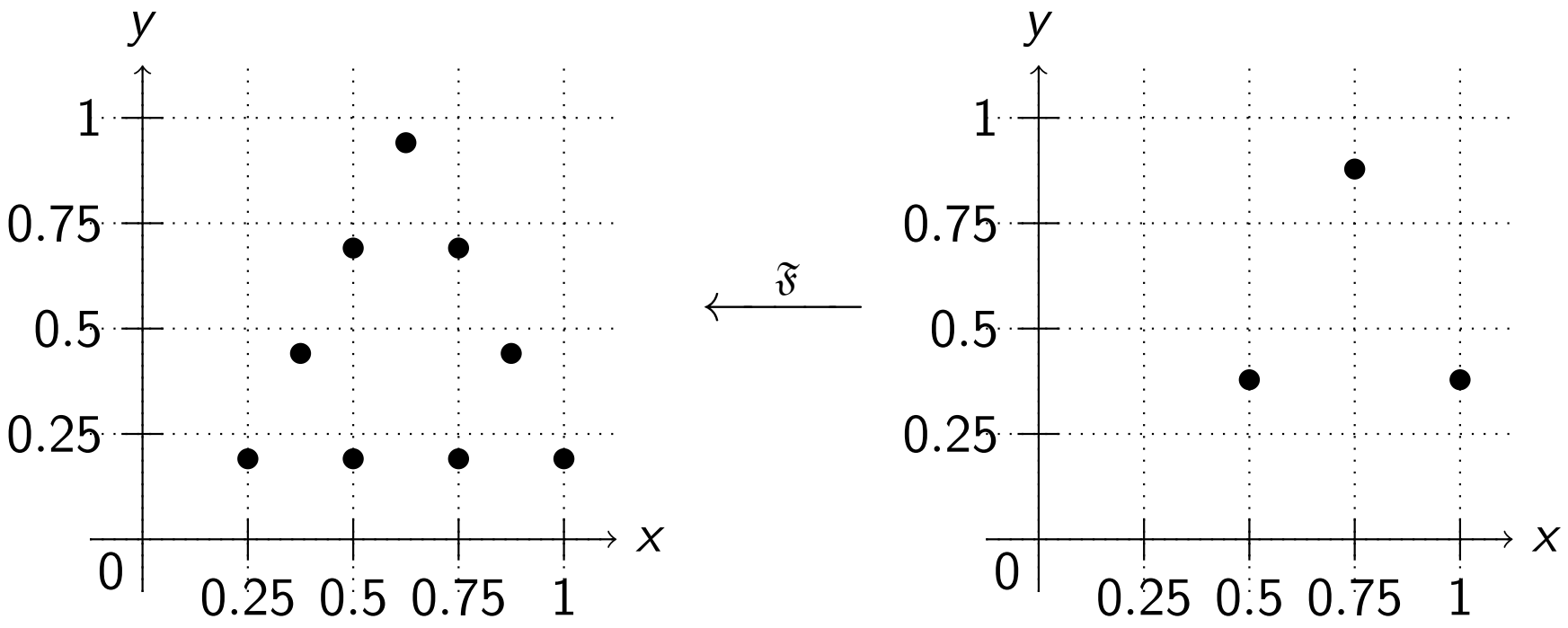
First iteration

Even if we start with a single point, we generate the subset $\mathfrak{F}(\{\bullet\}) := F_1(\{\bullet\}) \cup F_2(\{\bullet\}) \cup F_3(\{\bullet\})$ consisting of three points after the first iteration.



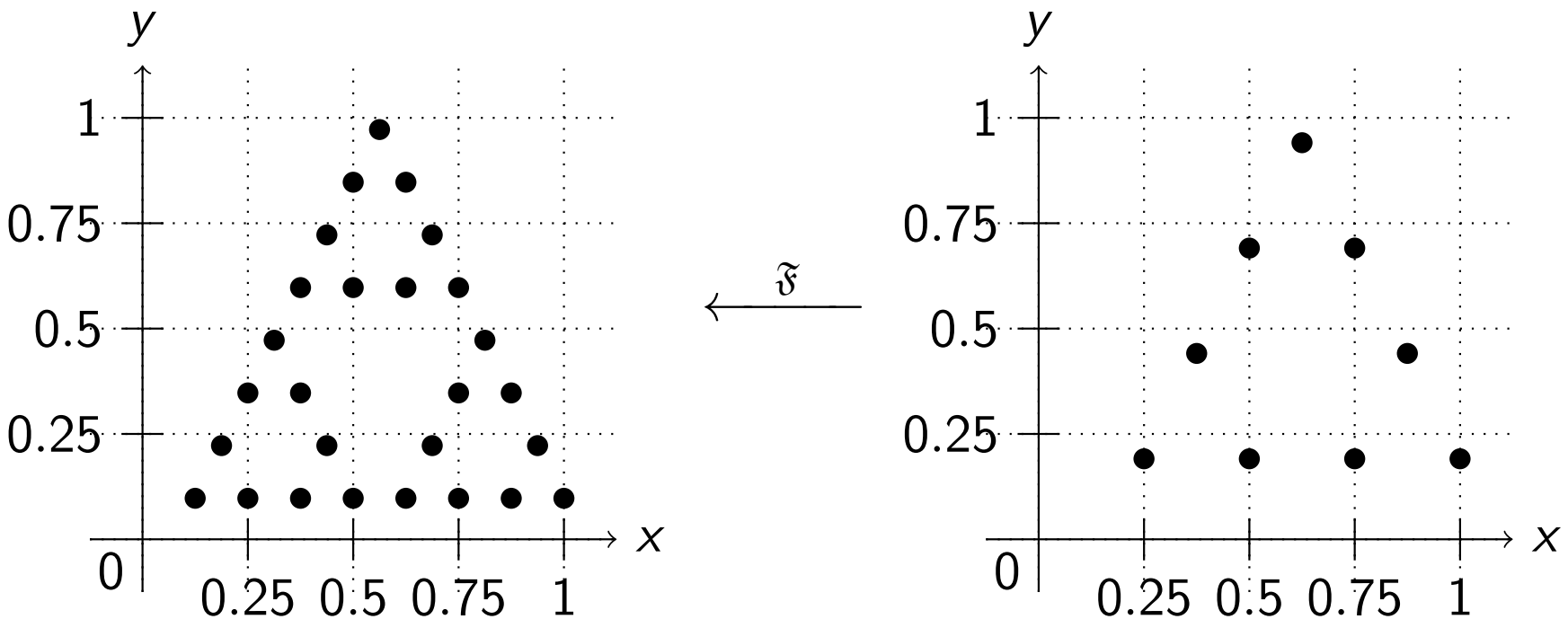
Second iteration

Now we apply $\mathfrak{F}$ to this subset consisting of three (3^{2-1}) points to obtain nine (3^{3-1}) points $\mathfrak{F}^2(\{\bullet\}) := \mathfrak{F}(\mathfrak{F}(\{\bullet\}))$.



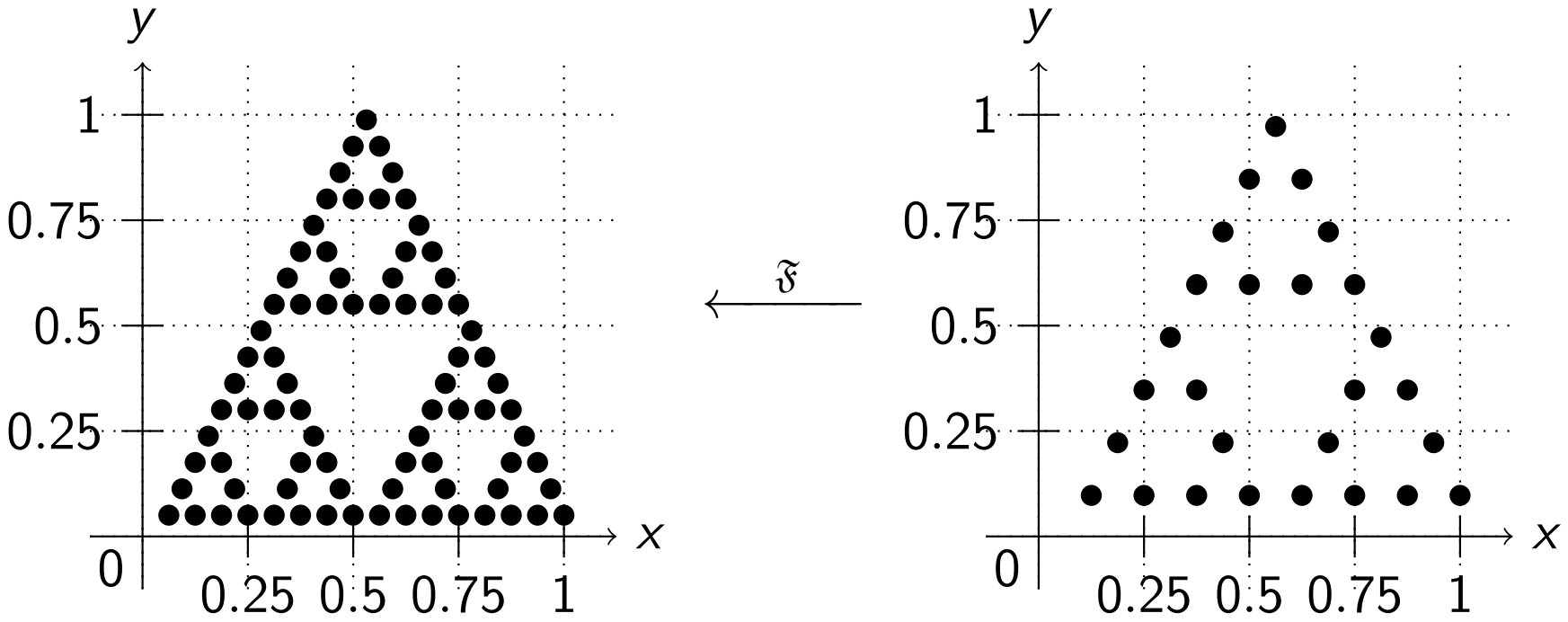
Third iteration

Now we apply $\mathfrak{F}$ to this subset consisting of nine (3^{3-1}) points to obtain $27 = 3^3$ points $\mathfrak{F}^3(\{\bullet\}) := \mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\{\bullet\})))$.



Fourth iteration

Now we apply $\mathfrak{F}$ to this subset consisting of $27 = 3^{4-1}$ points to obtain $81 = 3^4$ points $\mathfrak{F}^4(\{\bullet\}) := \mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\{\bullet\}))))$.



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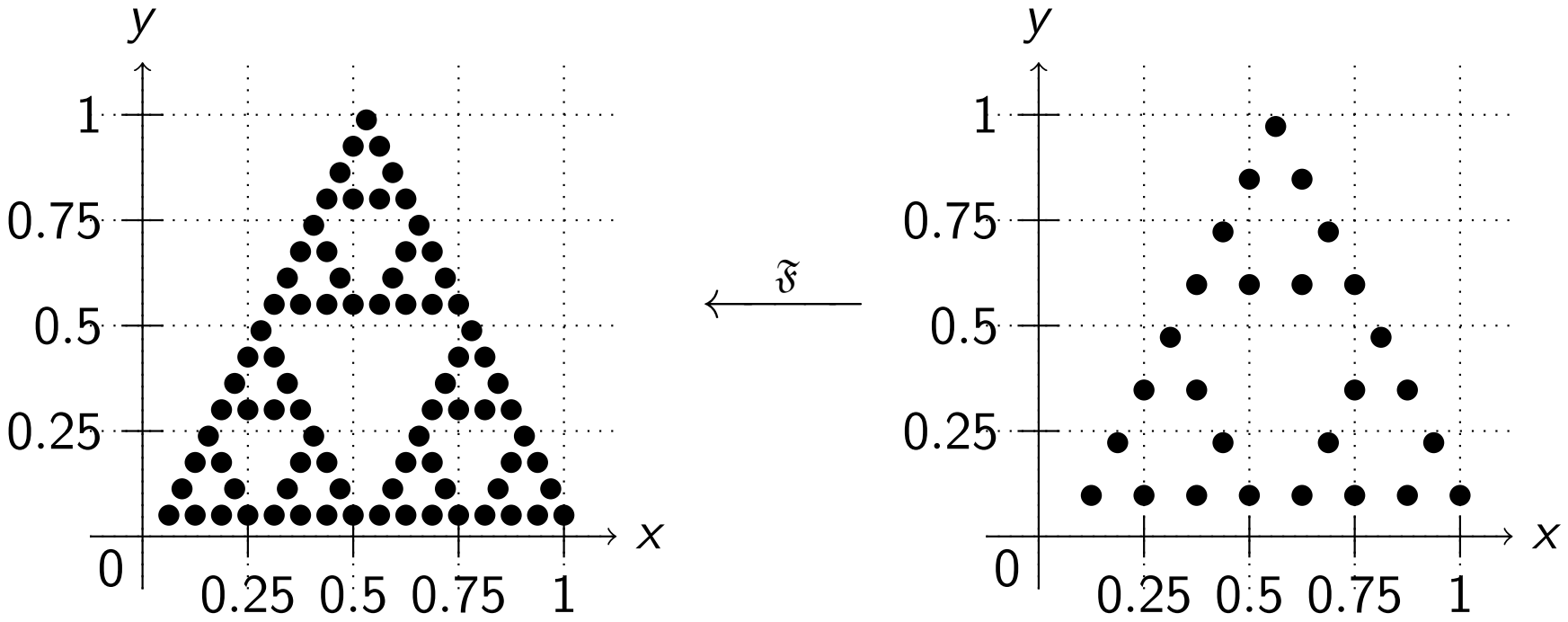
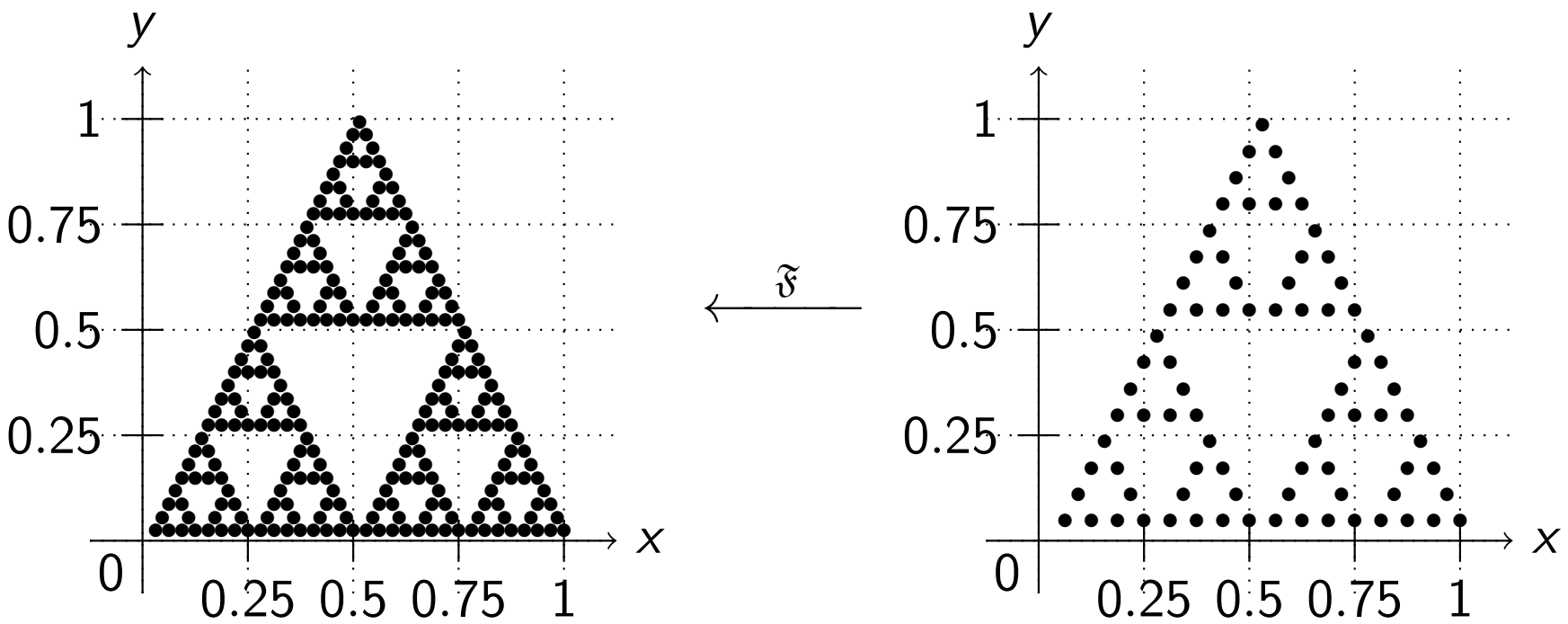


Our •'s are now a bit too big and might overlap on the next iteration, so we'll change their size in the next iteration.

Fifth iteration

Now we apply $\mathfrak{F}$ to this subset consisting of $81 = 3^{5-1}$ points to obtain $243 = 3^5$ points $\mathfrak{F}^5(\{\bullet\}) := \mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\{\bullet\})))))$.



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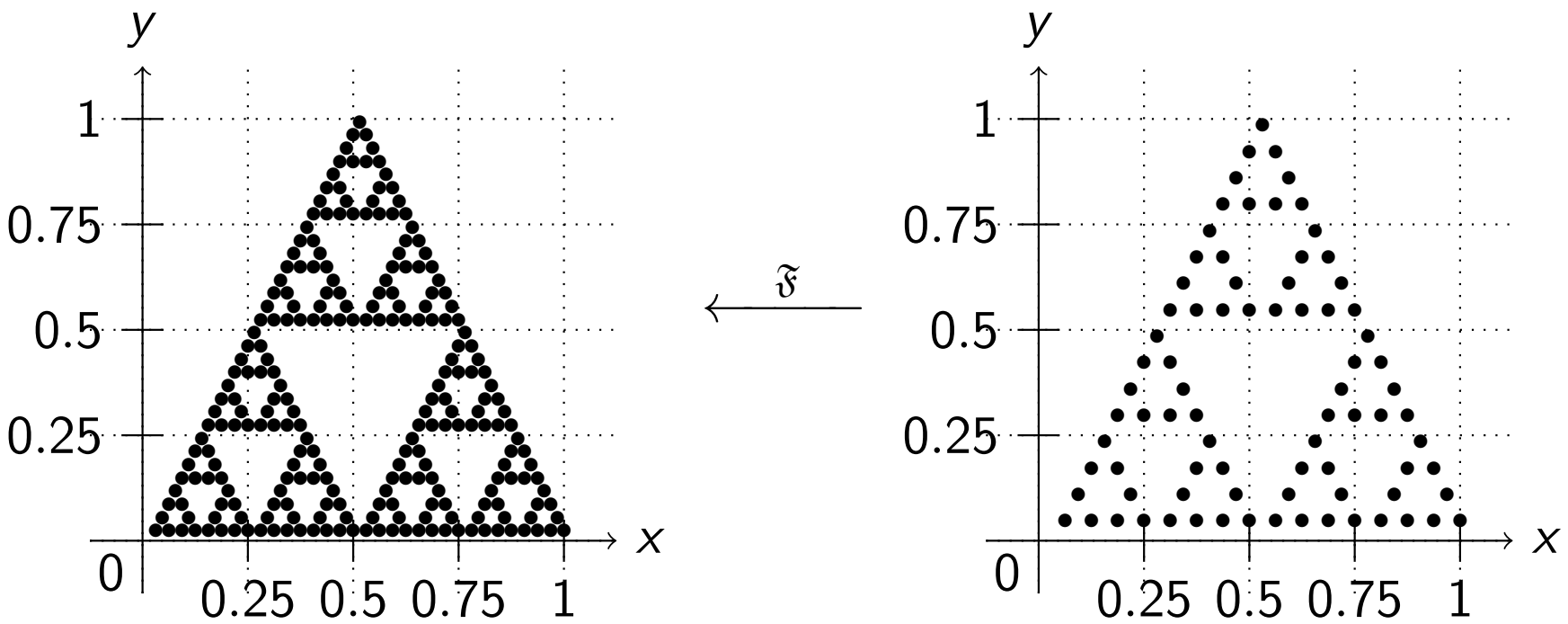
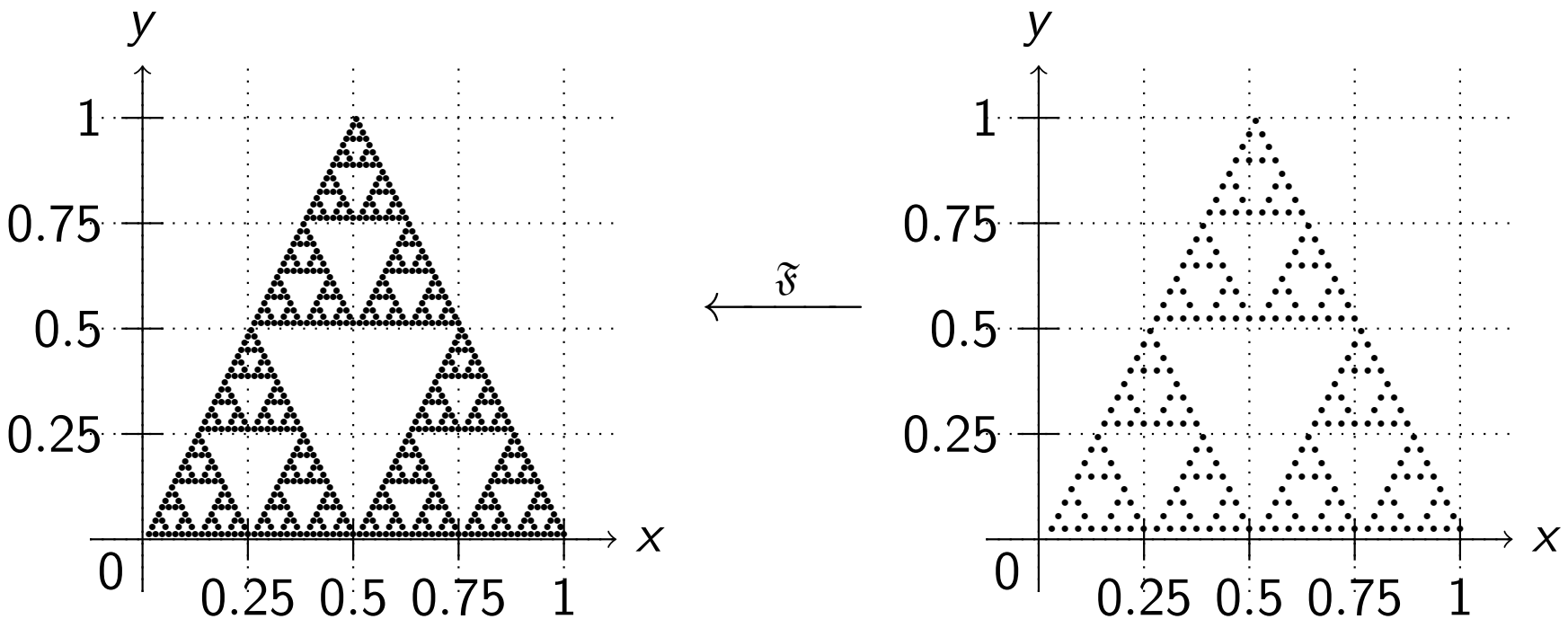


We have to make the •'s even smaller...

Sixth iteration

Now we apply $\mathfrak{F}$ to this subset consisting of $243 = 3^{6-1}$ points to obtain $729 = 3^6$ points $\mathfrak{F}^6(\{\bullet\}) := \mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\{\bullet\}))))))$.



Seventh iteration

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Are we seriously going to do this?

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Are we seriously going to do this? Yes...

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Are we seriously going to do this? Yes... Get ready for $2187 = 3^7$ points.

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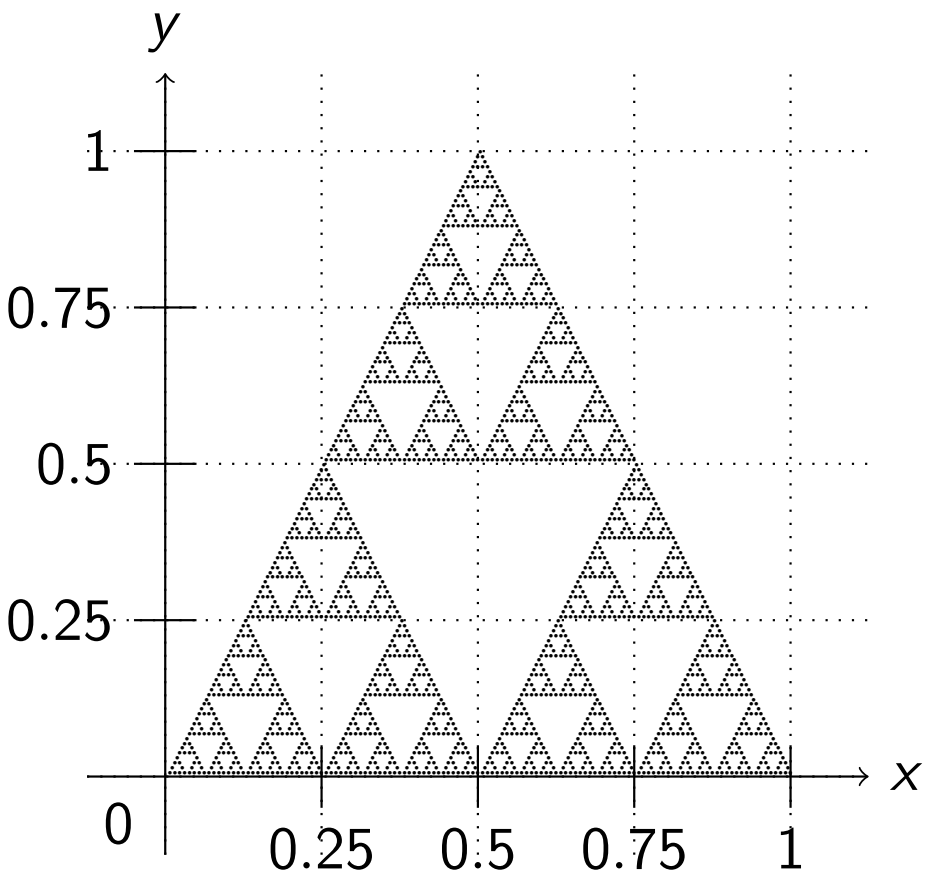
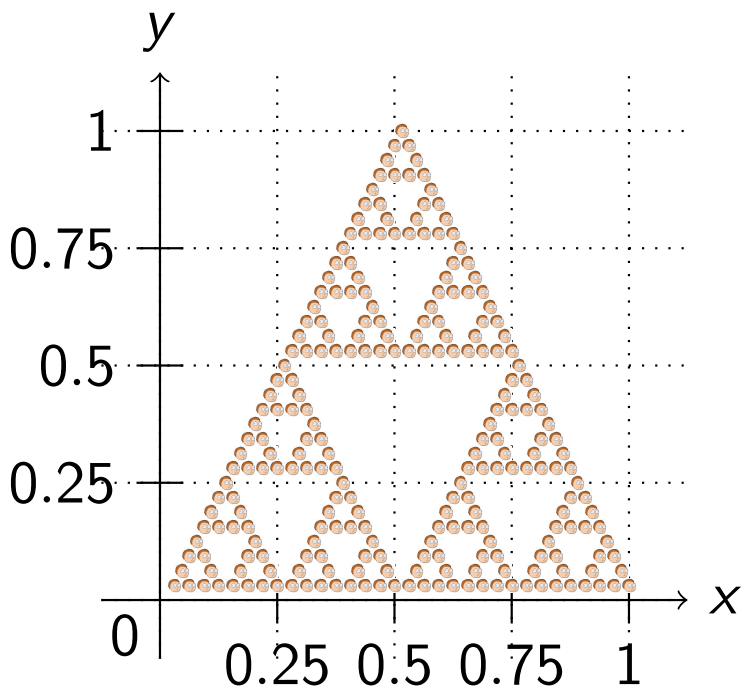
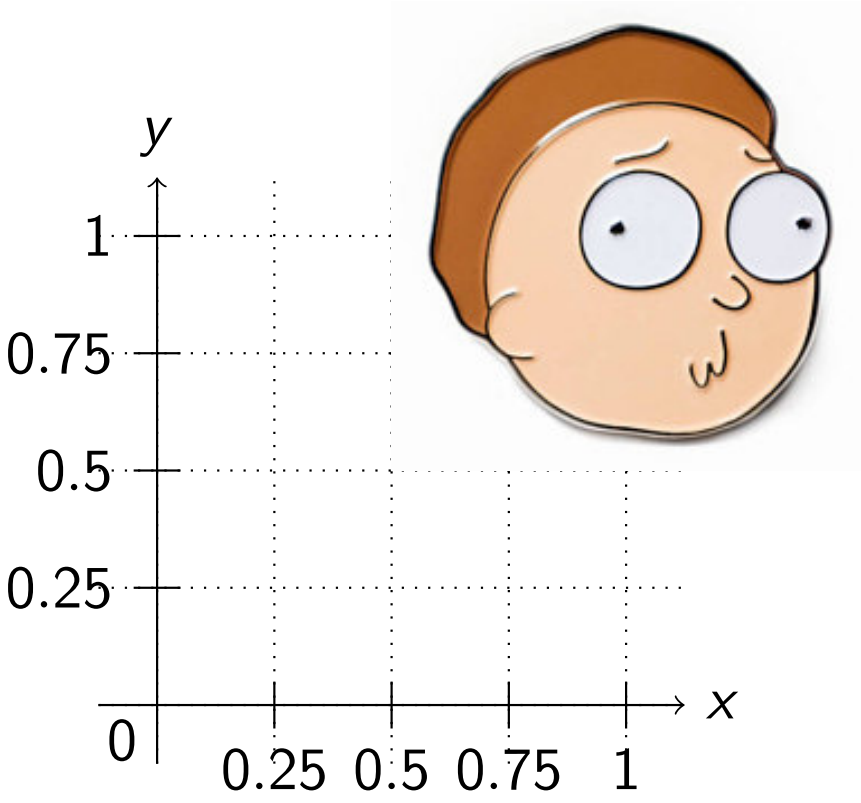


Invariance under initial conditions: subset

No matter what subset we start with initially, after we apply enough iterates of $\mathfrak{F}$, we seem to converge to the same thing!



$\xleftarrow{\mathfrak{F}^5}$



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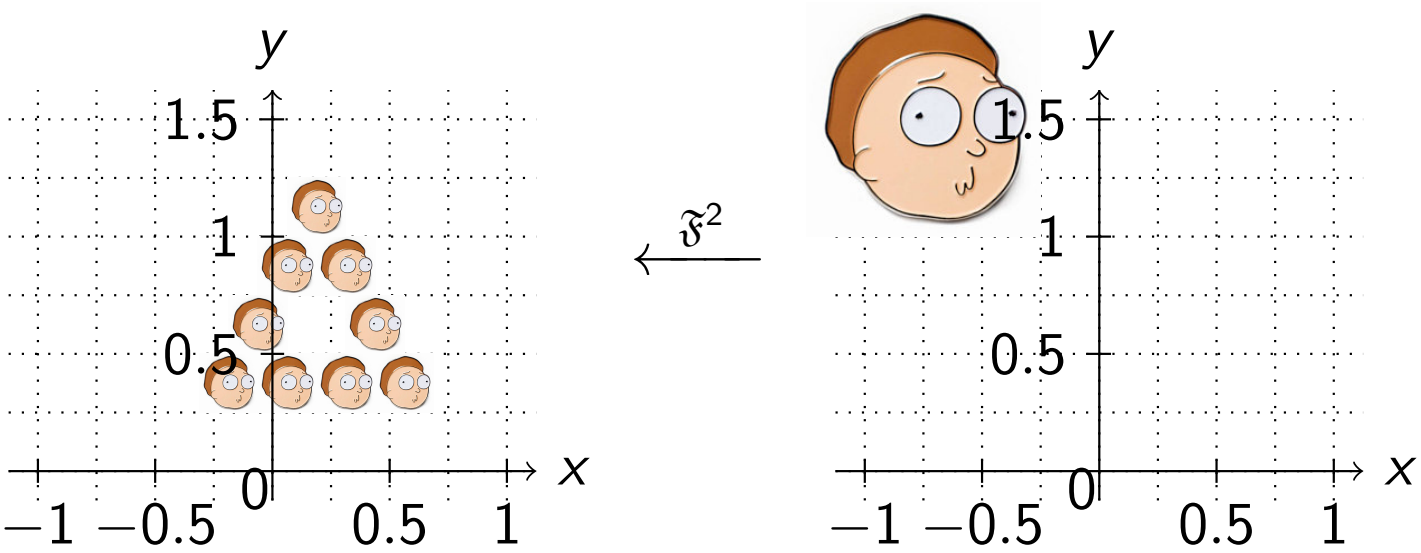

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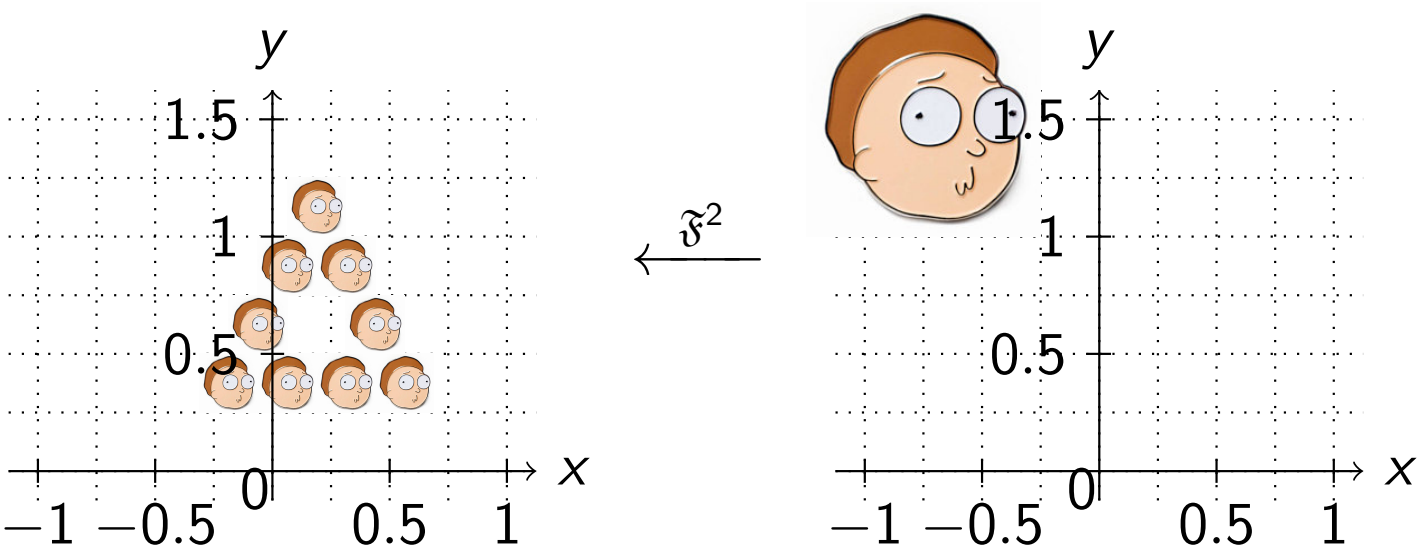

...Why?

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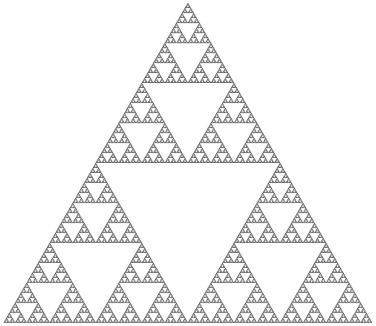
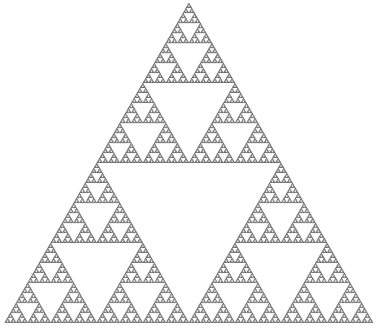
$\mathfrak{F}\Bigg($  $\Bigg) =$  .

The contraction mapping theorem

Theorem

Let (X, d) be a complete metric space and let $\mathfrak{F} : (X, d) \to (X, d)$ be a contraction. Then there exists a unique fixed point of $\mathfrak{F}$, i.e. a unique point $x \in X$ such that $\mathfrak{F}(x) = x$. In fact, for any $x_0 \in X$, the sequence

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In other words, no matter which closed and bounded subset of $\mathbb{R}^2$ we start with, applying $\mathfrak{F}$ repeatedly to this subset gets us arbitrarily close to the Sierpinski triangle.

The contraction mapping theorem in action

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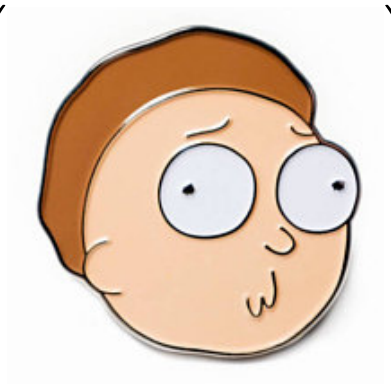

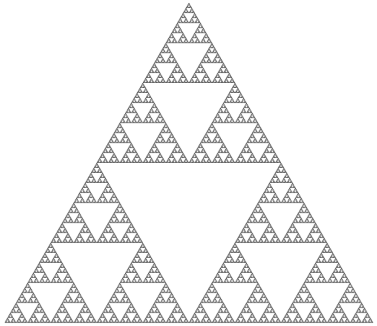

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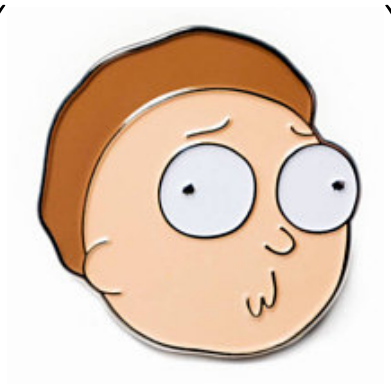

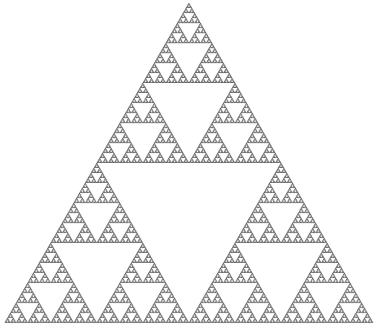

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Poor Morty...

What else can we apply the contraction mapping theorem to?

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- It can be used to prove the implicit function theorem from analysis.
- It is used to prove the existence and uniqueness of solutions to ordinary differential equations of a single variable.
- It is used to prove the existence and uniqueness of a steady state probability distribution for suitable Markov chains.

Differential equations I

Let me just mention my favorite one of these.

Theorem

Let $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function with the property that there exists an* L *such that*

$$\left|F(x,t) - F(y,t)\right| \leq L|x-y| \qquad \text{for all } x, y \in \mathbb{R} \text{ and } t \in [a,b].$$

Then the initial value problem (IVP)

$$\frac{dx(t)}{dt} = F(x,t) \qquad \text{with } x(a) \text{ specified}$$

has a unique solution.

Differential equations II

Idea of proof.

Define $\Phi_F : C\big([a,b]\big) \to C\big([a,b]\big)$ defined by sending a continuous function $x : [a,b] \to \mathbb{R}$ to the function $\Phi_F(x)$ whose value at $t \in [a,b]$ is given by

$$\big(\Phi_F(x)\big)(t) := x(a) + \int_a^t F\big(x(s), s\big)\, ds.$$

Notice that a solution x to the IVP satisfies $\Phi_F(x) = x$. Φ_F turns out to be a contraction on $C\big([a,b]\big)$ and the contraction mapping theorem can be applied. $\square$

Differential equations III

Notice that (being careful about notation is crucial here!)

$$\begin{aligned}\big(\Phi_F^2(x)\big)(t) &= \Big(\Phi_F\big(\Phi_F(x)\big)\Big)(t) \\ &= \big(\Phi_F(x)\big)(a) + \int_a^t F\Big(\big(\Phi_F(x)\big)(s), s\Big)\, ds \\ &= \big(\Phi_F(x)\big)(a) + \int_a^t F\left(x(a) + \int_a^s F(x,r)\, dr, s\right)\, ds\end{aligned}$$

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looks much more complicated than just $\Phi_F(x)$ alone. Nevertheless, by iterating Φ_F, we get close to the solution regardless of our initial guess for a function x. Φ_F updates our initial guess to a better guess.

Differential equations example I

To solve

$$\frac{dx}{dt} = (x+t)t \qquad \text{on } [0,1] \text{ with } x(0) = 0,$$

we first identify $F(x,t) = (x+t)t$.

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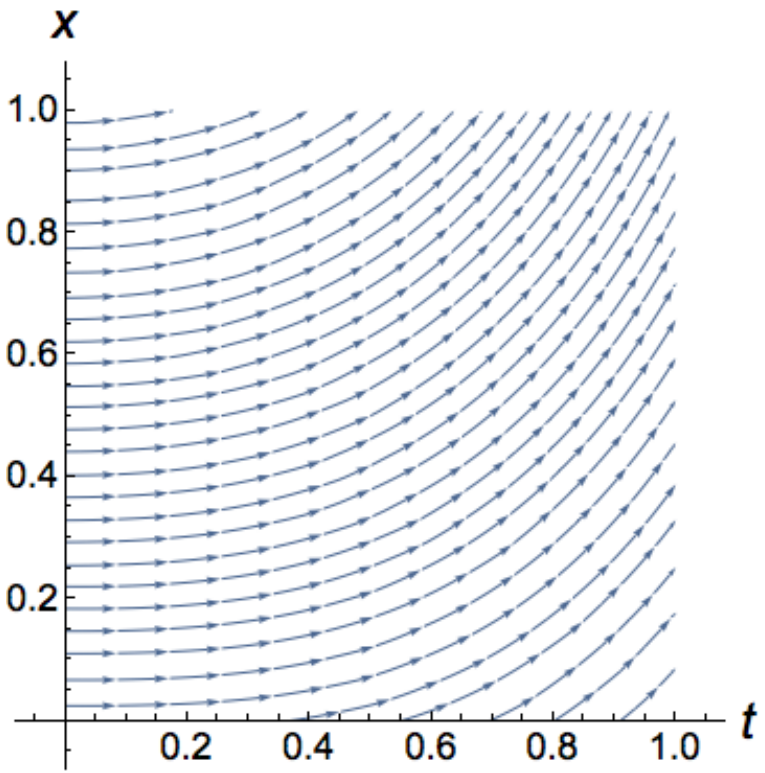
To solve

$$\frac{dx}{dt} = (x+t)t \qquad \text{on } [0,1] \text{ with } x(0) = 0,$$

we first identify $F(x,t) = (x+t)t$. If we plot the associated slope field, it looks like



Differential equations example II

Let us take our initial guess $x_0 = 0$, the function that is zero everywhere. Since $F(x,t) = (x+t)t$,

$$\big(\Phi_F(x)\big)(t) = x + \int_0^t (x+s)s\,ds.$$

Plugging in our initial guess gives

$$\big(\Phi_F(x_0)\big)(t) = 0 + \int_0^t (0+s)s\,ds = \int_0^t s^2\,ds = \frac{s^3}{3}.$$

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Applying this one more time gives

$$\begin{aligned}\big(\Phi_F^2(x_0)\big)(t) &= \left(\Phi_F\left(s \mapsto \frac{s^3}{3}\right)\right)(t) = \frac{0^3}{3} + \int_0^t F\left(\frac{s^3}{3}, s\right)\ ds \\ &= \int_0^t \left(\frac{s^3}{3} + s\right) s\ ds = \frac{t^5}{5\cdot 3} + \frac{t^3}{3}\end{aligned}$$

Differential equations example III

Applying this one more time gives

$$\begin{aligned}\left(\Phi_F^3(x_0)\right)(t) &= \left(\Phi_F\left(s \mapsto \frac{s^5}{5\cdot 3} + \frac{s^3}{3}\right)\right)(t) \\ &= \frac{0^5}{5\cdot 3} + \frac{0^3}{3} + \int_0^t F\left(\frac{s^5}{5\cdot 3} + \frac{s^3}{3}, s\right)\, ds \\ &= \int_0^t \left(\frac{s^5}{5\cdot 3} + \frac{s^3}{3} + s\right) s\, ds = \frac{t^7}{7\cdot 5\cdot 3} + \frac{t^5}{5\cdot 3} + \frac{t^3}{3}\end{aligned}$$

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$$\begin{aligned}\left(\Phi_F^3(x_0)\right)(t) &= \left(\Phi_F\left(s \mapsto \frac{s^5}{5\cdot 3} + \frac{s^3}{3}\right)\right)(t) \\ &= \frac{0^5}{5\cdot 3} + \frac{0^3}{3} + \int_0^t F\left(\frac{s^5}{5\cdot 3} + \frac{s^3}{3}, s\right)\, ds \\ &= \int_0^t \left(\frac{s^5}{5\cdot 3} + \frac{s^3}{3} + s\right) s\, ds = \frac{t^7}{7\cdot 5\cdot 3} + \frac{t^5}{5\cdot 3} + \frac{t^3}{3}\end{aligned}$$

From this, we can get the pattern and after infinitely many iterations we get

$$\lim_{n\to\infty}\left(\Phi_F^n(x_0)\right)(t) = \sum_{k=1}^{\infty} \frac{t^{2k+1}}{(2k+1)!!}$$

Differential equations example IV

Let's check to make sure that

$$x(t) = \sum_{k=1}^{\infty} \frac{t^{2k+1}}{(2k+1)!!}$$

is indeed a solution to $\frac{dx}{dt} = (x+t)t$ with initial condition $x(0) = 0$.

Differential equations example IV

Let's check to make sure that

$$x(t) = \sum_{k=1}^{\infty} \frac{t^{2k+1}}{(2k+1)!!}$$

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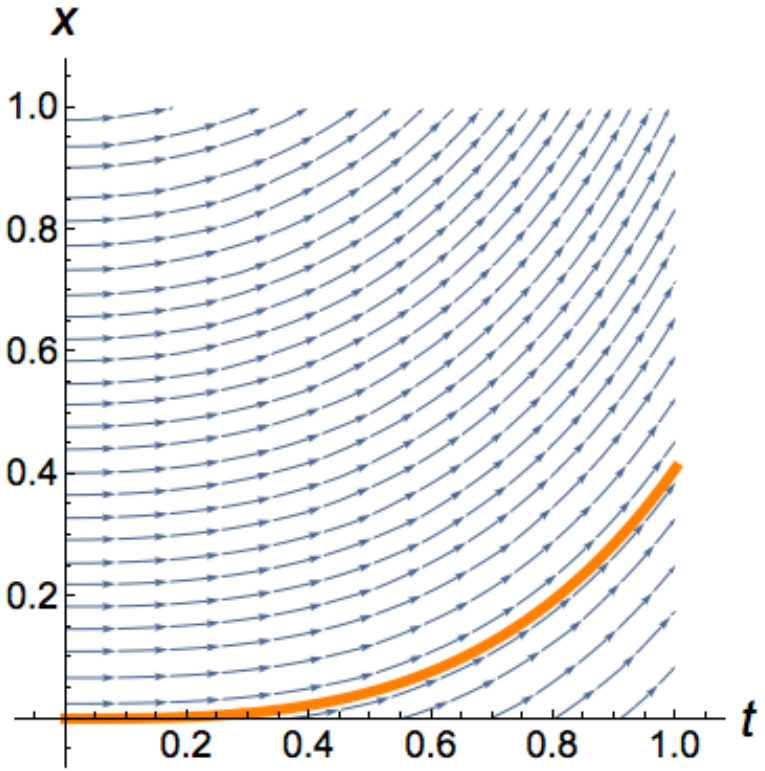

Thank you!

Thanks for your attention!
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2. Barnsley fern picture from DSP-user. Available at the wikipedia entry for the Barnsley Fern
3. John Hutchinson "Fractals and self similarity"
4. Bruno Lacroix "Fractal image compression"
5. My friend Link produced most of the Java code needed to plot the Sierpinski triangle
6. Keith Conrad provided some slides which I have found useful
7. The picture of Morty is from the show "Rick and Morty"
8. Victor Bryant "Metric spaces: Iteration and Application"