Two-dimensional algebra and gauge theory for strings
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All work presented here is based on "Gauge invariant surface holonomy and monopoles." Theory Appl. Categ., Vol. 30, 2015, No. 42, pp 1319-1428 (arXiv:1410.6938), references therein, and a forthcoming article.

1. Motivation

2. Groups and 2-groups
   - Groups and parallel transport
   - Two-dimensional algebra via 2-categories

3. Parallel transport for strings
   - Differential data for 2-form gauge fields
   - Infinitesimal surface transport
   - Local surface transport
   - Functorial properties of parallel transport
   - Gauge transformations
   - Gauge invariance and global surface transport

Motivation I

The Action for a charged particle of charge q and mass m moving along a trajectory $t \mapsto x(t)$ on a manifold with a metric (M, g) is (locally) given by

$$S[x, A] = -m \int \sqrt{g\left(\frac{dx}{dt}, \frac{dx}{dt}\right)}\, dt + q \int A\left(\frac{dx}{dt}\right) dt - \int_M F \wedge \star F$$

where $\star$ is the Hodge star operator, A is the 1-form electromagnetic potential, and $F := dA$ is the electromagnetic field strength. Notice that the interaction term (the middle term) does not depend on the metric (however, contrary to popular physics terminology, it is *not* "topological" since it depends on the smooth structure of M). This term is responsible for the Aharanov-Bohm effect as it arises as the phase factor

$$\exp\left\{ q \int A\left(\frac{dx}{dt}\right) dt \right\}.$$

Motivation II

This phase factor is also important in the case of non-abelian gauge theories where it appears as the path-ordered exponential

$$\mathcal{P}\exp\left\{q\int A\left(\frac{dx}{dt}\right)dt\right\}.$$

We will construct this more explicitly momentarily. The point is that these factors and their associated Wilson loops describe non-perturbative effects in gauge theory such as confinement as discovered by Wilson in 1974.

Motivation III

The analogous Action for strings coupled to an abelian gauge field B was conceived by Kalb and Ramond in 1974. Let h denote the metric on the string worldsheet x. Then the Action is

$$S[x,B] = -m \int \sqrt{-\det(h)} h^{ab} g\left(\partial_a x, \partial_b x\right)\, dsdt$$
$$+ q \int B\left(\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}\right) dsdt - \int_M H \wedge \star H$$

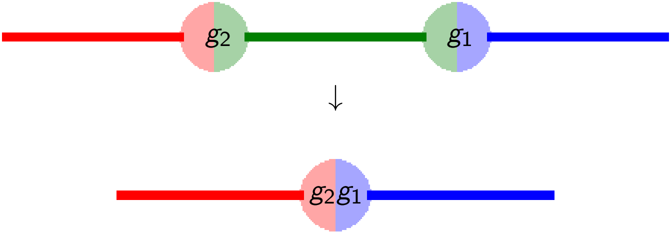
where H is the electromagnetic field strength 3-form $H := dB$. In this talk, we will focus on a non-abelian generalization of the interaction term for strings and its appropriate exponential

$$\exp\left\{ q \int B\left(\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}\right) dsdt \right\}$$

in order to make sense of Wilson surfaces, which are analogues of Wilson loops for particles.

Groups are categories I

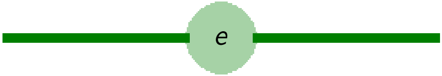
A group can be thought of as a set with an associative binary operation, an identity for the operation, and an inverse for each element. Categories are like groups with *partially* defined associative binary operations. You can only multiply elements if their "colors" match:



The lines are 1-dimensional "domains" and the bullets are 0-dimensional "defects."

Groups are categories II

Every color has an identity



Groups are examples where all the colors are the same, i.e. all the domains are indistinguishable. Therefore, groups are examples of categories.

Groups describe parallel transport I

The solution to the initial value problem differential equation

$$\frac{d\psi(t)}{dt} = A(t)\psi(t), \qquad \psi(0) \equiv \psi_0 \in \mathbb{R}^n$$

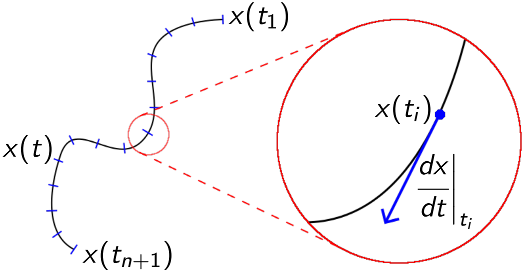
with $A(t)$ a time-dependent $n \times n$ matrix is

$$\psi(t) = \psi_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^t dt_k \cdots \int_0^t dt_1 \; \mathcal{T}\left[A(t_k) \cdots A(t_1)\right] \psi_0$$

where $\mathcal{T}$ stands for time-ordering with earlier times appearing to the right. This shows up in several contexts such as (a) solving Schrödinger's equation with $A(t) = -iH(t)$ for a time-dependent Hamiltonian and ψ a vector in the space on which H acts and (b) calculating the parallel transport along a curve in gauge theory, where A is the local vector potential. This integral goes under many names: Dyson series, Picard iteration, path/time-ordered exponential, etc.

Groups describe parallel transport II

Infinitesimally, one can imagine the solution to this differential equation as coming from breaking up a curve into infinitesimal paths



and associating the group elements

$$\exp\left\{A_{\mu_i}(x(t_i))\left.\frac{dx^{\mu_i}}{dt}\right|_{t_i}\right\} \cong 1 + A_{\mu_i}(x(t_i))\left.\frac{dx^{\mu_i}}{dt}\right|_{t_i}$$

at these paths and multiplying those group elements in the order dictated by the path. By locality, the group elements should be of this form.

Groups describe parallel transport III

Keeping the order, the result of multiplying all these elements is

$$\exp\left\{A_{\mu_n}(x(t_n))\frac{dx^{\mu_n}}{dt}\bigg|_{t_n}\right\}\cdots\exp\left\{A_{\mu_1}(x(t_1))\frac{dx^{\mu_1}}{dt}\bigg|_{t_1}\right\}.$$

Expanding out to lowest order (since the paths are infinitesimal) gives

$$\left(1+A_{\mu_n}(x(t_n))\frac{dx^{\mu_n}}{dt}\bigg|_{t_n}\right)\cdots\left(1+A_{\mu_1}(x(t_1))\frac{dx^{\mu_1}}{dt}\bigg|_{t_1}\right)$$

and reorganizing terms results in

$$1+\sum_{i=1}^{n}A_{\mu_i}(x(t_i))\frac{dx^{\mu_i}}{dt}\bigg|_{t_i}+\sum_{i\geq j\geq 1}^{n}A_{\mu_i}(x(t_i))A_{\mu_j}(x(t_j))\frac{dx^{\mu_i}}{dt}\bigg|_{t_i}\frac{dx^{\mu_j}}{dt}\bigg|_{t_j}+\cdots,$$

which is exactly the path-ordered integral we saw before (after taking the $n\to\infty$ limit).

Groups describe parallel transport IV

We picture this group element as all the number of ways in which A interacts with the particle preserving the order of the worldline

$$1$$

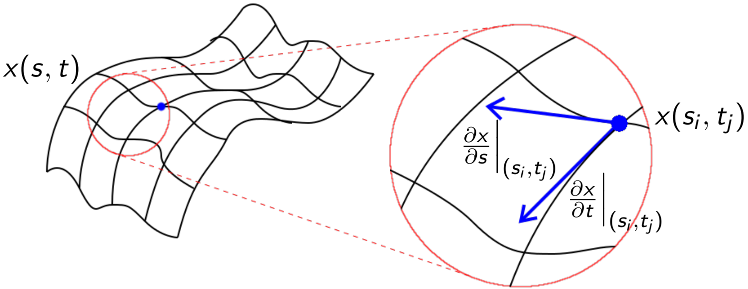
$$+\int A_{\mu_1}(t_1)\frac{dx^{\mu_1}}{dt}\bigg|_{t_1} \quad dt_1$$

$$+\int\int A_{\mu_2}(t_2)\frac{dx^{\mu_2}}{dt}\bigg|_{t_2} \quad A_{\mu_1}(t_1)\frac{dx^{\mu_1}}{dt}\bigg|_{t_1} \quad dt_1 dt_2$$

$$+\cdots$$

Groups do not describe parallel transport along surfaces I

But what if we had a string? We would like to couple a non-abelian 2-form B and attempt a similar procedure



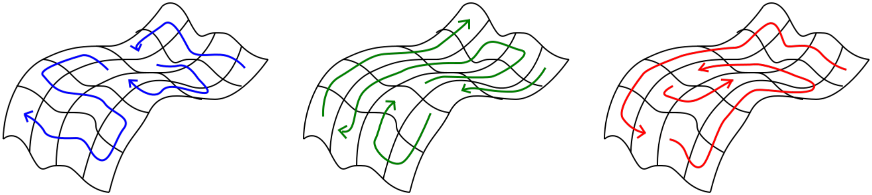
associating the group elements (again this follows from locality)

$$\exp\left\{ B_{\mu_i\mu_j}(x(s_i,t_j)) \frac{\partial x^{\mu_i}}{\partial s}\frac{\partial x^{\mu_j}}{\partial t}\Big|_{(s_i,t_j)} \right\}$$

to plaquettes.

Groups do not describe parallel transport along surfaces II

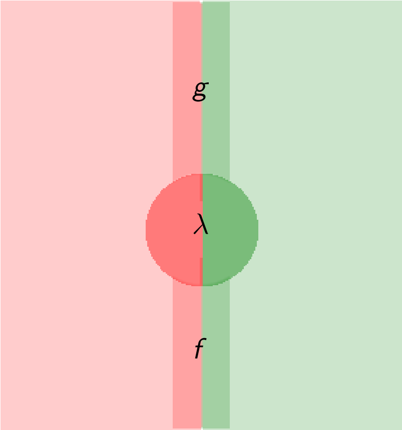
But then in which order should we multiply these elements?



Even if we chose a consistent way, by changing the parametrization on the worldsheet, the order would be different. This lead Teitelboim to conclude that only a $U(1)$ gauge field can couple to the worldsheet. However, there is a way around this problem using 2-category theory and the introduction of lower form gauge fields.

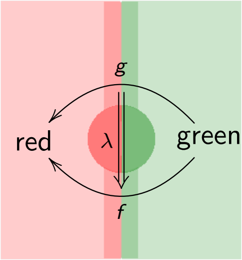
2-categories I

2-categories have 2-dimensional (2-d) domains, 1-dimensional (1-d) defects/domains between 2-d domains, and 0-dimensional (0-d) defects between the 1-d defects/domains.



2-categories II

The above depiction is related to the usual presentation of 2-categories via

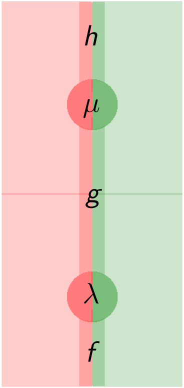


and are called "string diagrams." 1-d defects can be composed/fused horizontally (aka "in parallel")

Every 2-d domain has an identity 1-d defect.

2-categories III

0-d defects can be composed/fused vertically (aka "in series")



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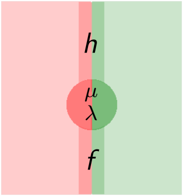
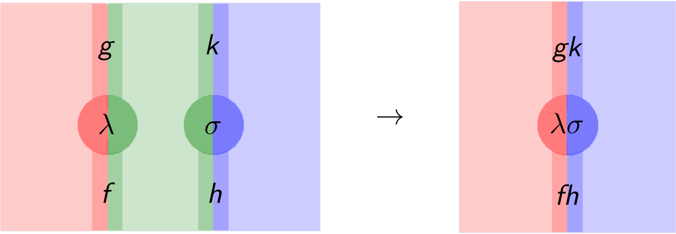


Note that the 1-d defect labelled by g must match. Every 1-d defect/domain has an identity 0-d defect for the vertical composition.

2-categories IV

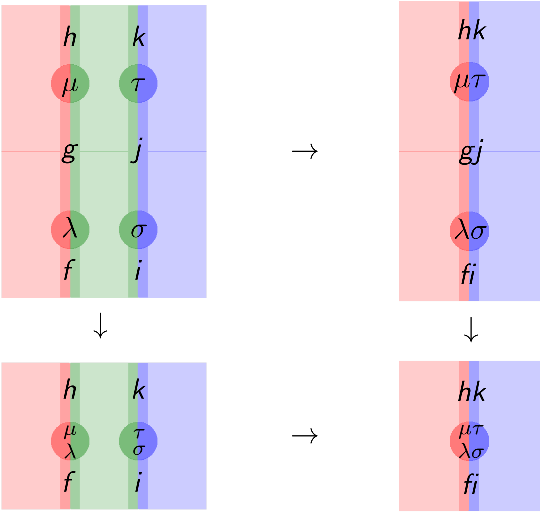
0-d defects can be also be composed/fused horizontally (aka "in parallel")



Every 2-d domain has an identity 0-d defect (which is the identity 0-d defect of the identity 1-d defect) for the horizontal composition.

2-categories V

The compositions must satisfy the “interchange law”



2-Groups and crossed modules I

A *2-group* is a 2-category all of whose two-dimensional domains are identical and all defects are invertible with respect to all compositions. Crossed modules allow us to be a little more concrete. A *crossed module* is a quadruple (H, G, τ, α) of two groups, G and H, group homomorphisms $\tau : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$, satisfying the two conditions

$$\alpha_{\tau(h)}(h') = hh'h^{-1}$$

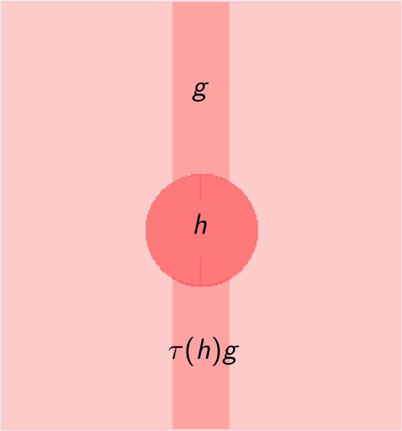
and

$$\tau(\alpha_g(h)) = g\tau(h)g^{-1}.$$

If the groups G and H are Lie groups and the maps τ and α are smooth, then (H, G, τ, α) is called a *Lie crossed module*. Before giving concrete examples, we show how crossed modules give rise to 2-groups.

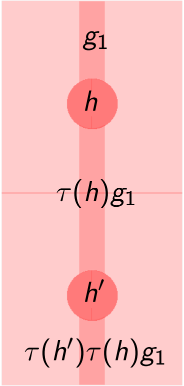
2-Groups and crossed modules II

1-dimensional defects are labelled by elements g of G. 0-dimensional defects are labelled by elements h of H. However, the labelling is not arbitrary and must be of the form

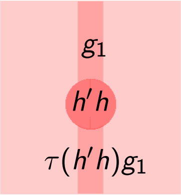


2-Groups and crossed modules III

The vertical composition is defined by

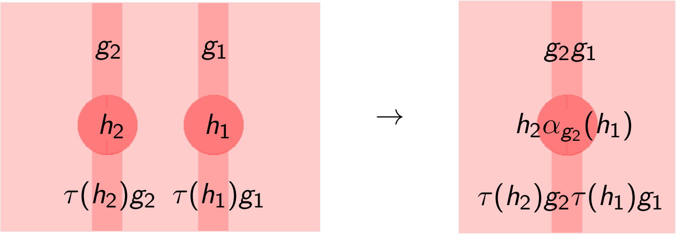


$\to$



2-Groups and crossed modules IV

The horizontal composition is defined by



Notice that

$$\tau(h_2)g_2\tau(h_1)g_1 = \tau\Big(h_2\alpha_{g_2}(h_1)\Big)g_2g_1.$$

The interchange law follows from the two identities. Big (but easy to prove) theorem: every 2-group arises in this way for some crossed module.

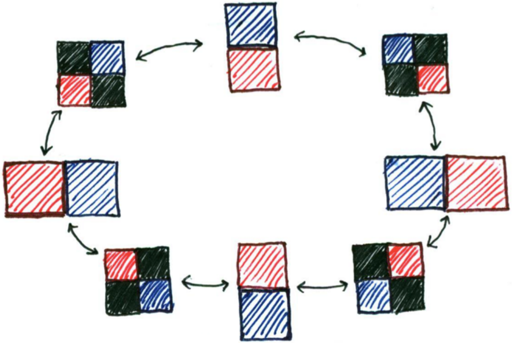
Examples of crossed modules I

Remember, a crossed module consists of (H, G, τ, α) with $\tau : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$.

- Let G be any group, $H := G$, $\tau := \mathrm{id}_G$, and α is conjugation.
- Let H be any group, $G := \mathrm{Aut}(H)$, $\tau(h)$ is the automorphism defined by $\tau(h)(h') := hh'h^{-1}$, and $\alpha := \mathrm{id}_{\mathrm{Aut}(H)}$.
- Let N be a normal subgroup of G. Set $H := N$, τ the inclusion, and α conjugation.
- Let G be a Lie group, $\tau : H \to G$ a covering space, and α conjugation by a lift.
- Let $G := \{*\}$, the trivial group, H any *abelian* group, τ the trivial map, and α the trivial map.

Examples of crossed modules II

Warning: It is *not* possible for H to be a non-abelian group if G is trivial! In fact, for an arbitrary crossed module (H, G, τ, α), $\ker(\tau)$ is always a central subgroup of H. This is in fact what Teitolboim observed and the following picture in terms of domains and defects illustrates a cute proof (red and blue indicate two different elements of H and black indicates the identity of H).



Differential data for 2-connections

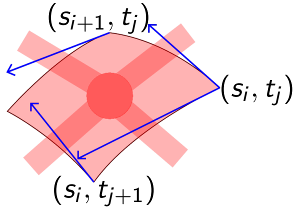
Now we come back to trying to formulate a consistent way to obtain non-abelian phase factors via parallel transport along surfaces.

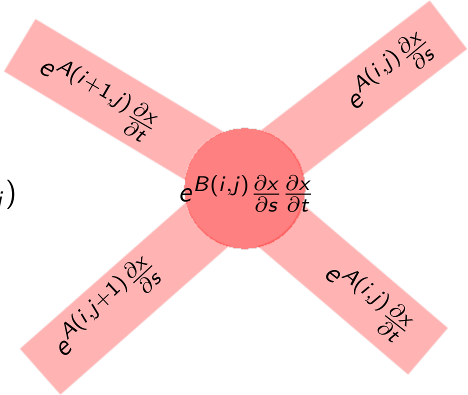
Let (H, G, τ, α) be a Lie crossed module and $(\mathfrak{h}, \mathfrak{g}, \underline{\tau}, \underline{\alpha})$ the associated Lie algebra data. The differential cocycle data for a trivial principal 2-bundle with connection over a smooth manifold M consists of a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(M; \mathfrak{g}))$, and a $\mathfrak{h}$-valued 2-form $B \in \Omega^2(M; \mathfrak{h}))$ satisfying

$$dA + \frac{1}{2}[A, A] = \underline{\tau}(B).$$

Infinitesimal surface transport I

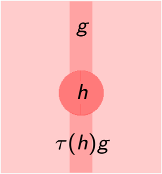
Therefore, what we should do is associate to each infinitesimal square spanned by the two coordinate vectors on our worldsheet the following group elements





Infinitesimal surface transport II

By the rules of 2-groups



the diagram requires that (at least to lowest order)

$$
\tau\Bigg(\overbrace{\exp\left\{B_{\mu_i\mu_j}(x(s_i,t_j))\frac{\partial x^{\mu_i}}{\partial s}\frac{\partial x^{\mu_j}}{\partial t}\bigg|_{(s_i,t_j)}\right\}}^{h}\Bigg)
$$

$$
\times\overbrace{\exp\left\{A_{\mu_j}(x(s_{i+1},t_j))\frac{\partial x^{\mu_j}}{\partial t}\bigg|_{(s_{i+1},t_j)}\right\}\exp\left\{A_{\mu_i}(x(s_i,t_j))\frac{\partial x^{\mu_i}}{\partial s}\bigg|_{(s_i,t_j)}\right\}}^{g}
$$

$$
=\exp\left\{A_{\mu_i}(x(s_i,t_{j+1}))\frac{\partial x^{\mu_i}}{\partial s}\bigg|_{(s_i,t_{j+1})}\right\}\exp\left\{A_{\mu_j}(x(s_i,t_j))\frac{\partial x^{\mu_j}}{\partial t}\bigg|_{(s_i,t_j)}\right\}.
$$

Infinitesimal surface transport III

But this follows from the standard calculation that (to lowest order) the holonomy around an infinitesimal loop is unity plus the curvature

$$\exp\left\{A_{\mu_i}(x(s_i,t_{j+1}))\frac{\partial x^{\mu_i}}{\partial s}\Big|_{(s_i,t_{j+1})}\right\}\exp\left\{A_{\mu_j}(x(s_i,t_j))\frac{\partial x\mu_j}{\partial t}\Big|_{(s_i,t_j)}\right\}$$

$$\times\exp\left\{-A_{\mu_i}(x(s_i,t_j))\frac{\partial x^{\mu_i}}{\partial s}\Big|_{(s_i,t_j)}\right\}\exp\left\{-A_{\mu_j}(x(s_{i+1},t_j))\frac{\partial x^{\mu_j}}{\partial t}\Big|_{(s_{i+1},t_j)}\right\}$$

$$=1+F_{\mu_i\mu_j}\big(x(s_i,t_j)\big)\frac{\partial x^{\mu_i}}{\partial s}\frac{\partial x^{\mu_j}}{\partial t}\Big|_{(s_i,t_j)},$$

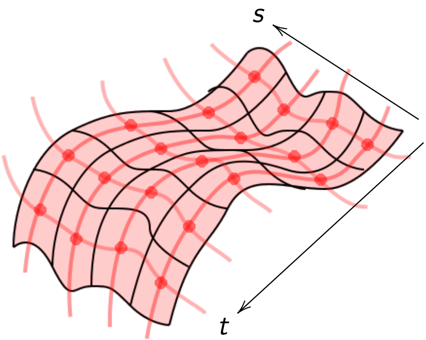
where F is the curvature of A, i.e. $F=dA+\frac{1}{2}[A,A]$, and the fact that

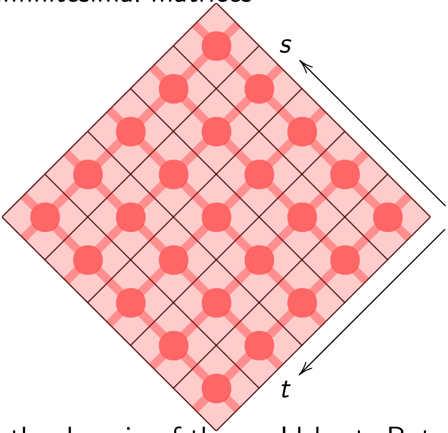
$$\tau\left(\exp\left\{B_{\mu_i\mu_j}\big(x(s_i,t_j)\big)\frac{\partial x^{\mu_i}}{\partial s}\frac{\partial x^{\mu_j}}{\partial t}\Big|_{(s_i,t_j)}\right\}\right)$$

$$=1+\underline{\tau}\left(B_{\mu_i\mu_j}\big(x(s_i,t_j)\big)\frac{\partial x^{\mu_i}}{\partial s}\frac{\partial x^{\mu_j}}{\partial t}\Big|_{(s_i,t_j)}\right).$$

Local surface transport I

Using our rules and the infinitesimal group element associated to each plaquette, we can piece together the infinitesimal matrices

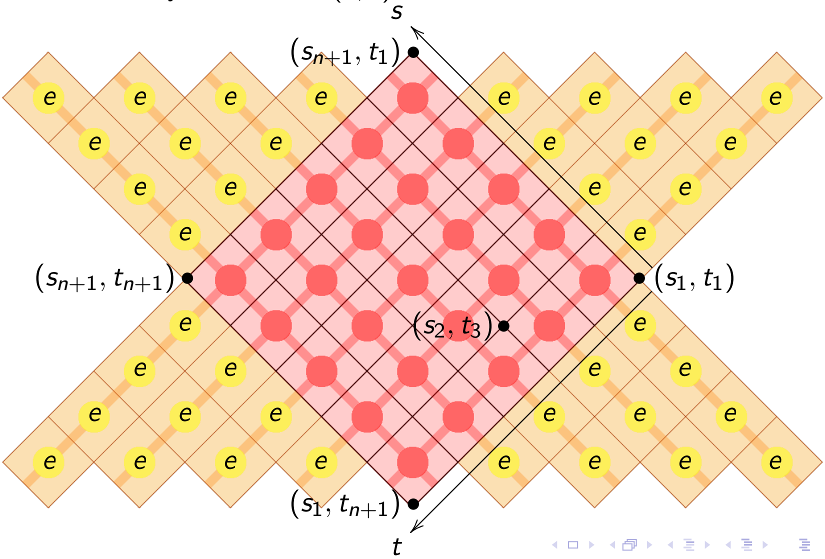




and simplify the picture drawing it on the domain of the worldsheet. But in order to multiply using our rules from before, we need to “fill in” some empty slots with identity 0-d defects.

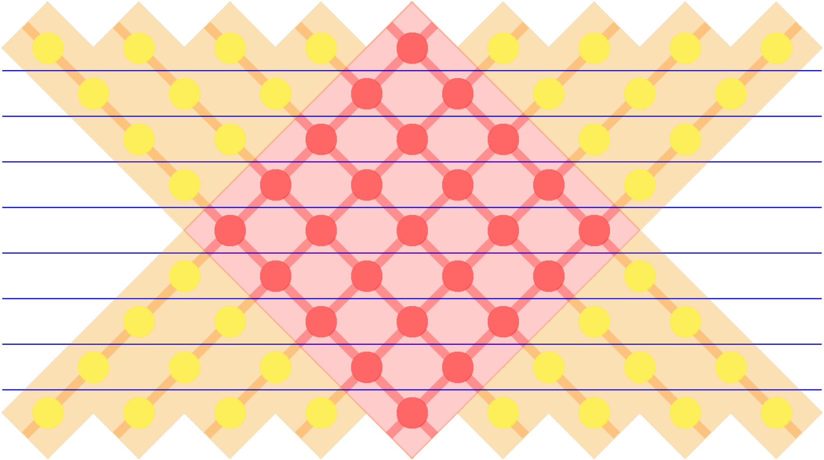
Local surface transport II

So let's do exactly this on the (s,t) domain of the worldsheet



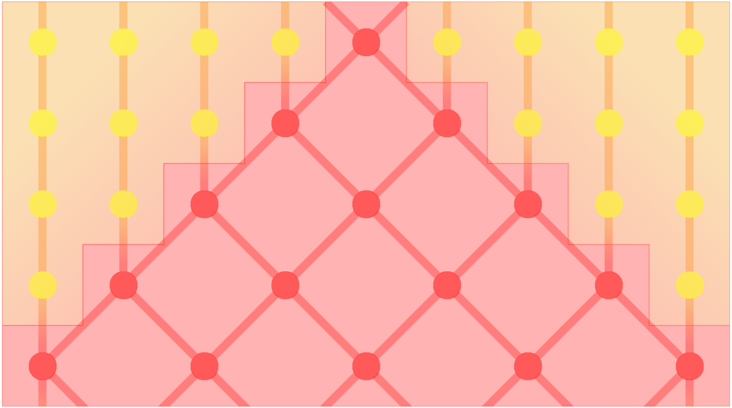
Local surface transport III

Filling in the identities allows us to horizontally compose the elements in the following way



Local surface transport IV

We can draw this in a more familiar way by tilting the elements (only the top half is drawn)



which now makes it easy to see we can first horizontally compose each row and then vertically compose the results in the remaining column.

Local surface transport V

We now need to label the 1-d and 0-d defects appropriately using our infinitesimal rules. We use the shorthand notation

$$a^s_{ij} := \exp\left\{ A\big(x(s_i,t_j)\big)\frac{\partial x}{\partial s}\Big|_{(s_i,t_j)} \right\}$$

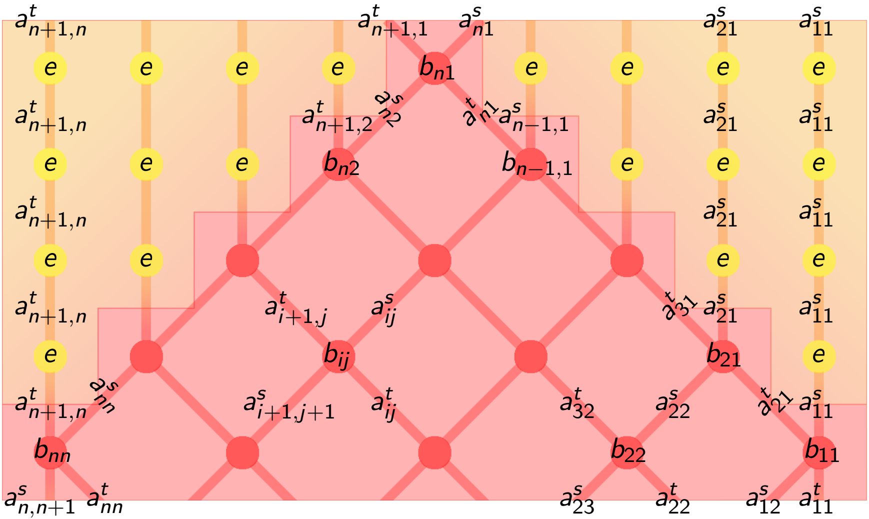
$$a^t_{ij} := \exp\left\{ A\big(x(s_i,t_j)\big)\frac{\partial x}{\partial t}\Big|_{(s_i,t_j)} \right\}$$

to denote the parallel transport along infinitesimal paths and

$$b_{ij} := \exp\left\{ B\big(x(s_i,t_j)\big)\frac{\partial x}{\partial s}\frac{\partial x}{\partial t}\Big|_{(s_i,t_j)} \right\}$$

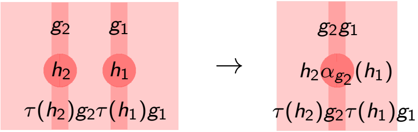
to denote the parallel transport along infinitesimal squares. A sum over components in the coordinates is assumed in each argument of the exponential. Then, our diagram becomes

Local surface transport VI



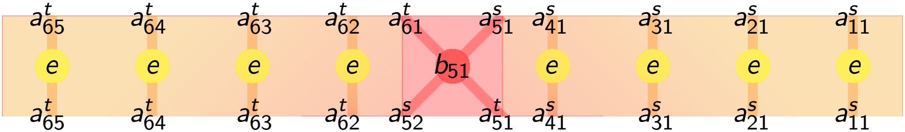
Local surface transport VII

Taking the first row and horizontally composing (recall, the rule for horizontal composition is:



)

gives

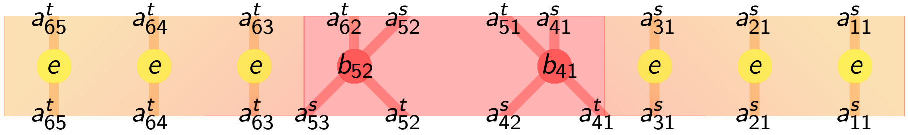


The result on the 1-d defects is just the usual group multiplication product while the result on the 0-d defects is

$$\alpha_{a^t_{65}a^t_{64}a^t_{63}a^t_{62}}(b_{51}).$$

Local surface transport VIII

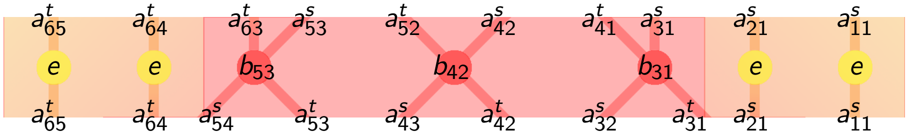
Taking the second row and horizontally composing gives



The result of composing the 0-d defects is

$$\alpha_{a^t_{65}a^t_{64}a^t_{63}}(b_{52})\alpha_{a^t_{65}a^t_{64}a^t_{63}a^t_{62}a^s_{52}}(b_{41}).$$

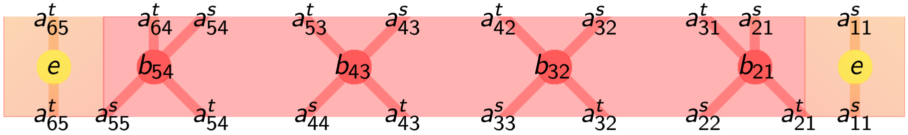
The third row gives



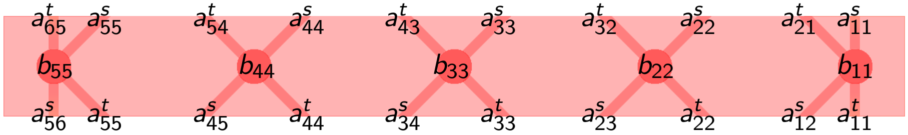
$$= \alpha_{a^t_{65}a^t_{64}}(b_{53})\alpha_{a^t_{65}a^t_{64}a^t_{63}a^s_{53}}(b_{42})\alpha_{a^t_{65}a^t_{64}a^t_{63}a^s_{53}a^t_{52}a^s_{42}}(b_{31}).$$

Local surface transport IX

The rest of the rows give



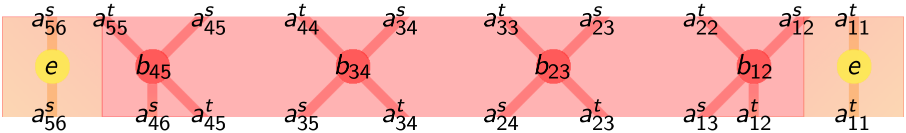
$$= \alpha_{a^t_{65}}\big(b_{54}\big)\alpha_{a^t_{65}a^t_{64}a^s_{54}}\big(b_{43}\big)\alpha_{a^t_{65}a^t_{64}a^s_{54}a^t_{53}a^s_{43}}\big(b_{32}\big)\alpha_{a^t_{65}a^t_{64}a^s_{54}a^t_{53}a^s_{43}a^t_{42}a^s_{32}}\big(b_{21}\big)$$



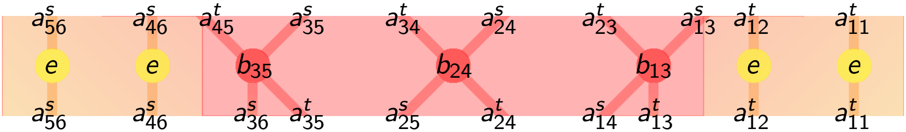
$$= b_{55}\alpha_{a^t_{65}a^s_{55}}\big(b_{44}\big)\alpha_{a^t_{65}a^s_{55}a^t_{54}a^s_{44}}\big(b_{33}\big)\alpha_{a^t_{65}a^s_{55}a^t_{54}a^s_{44}a^t_{43}a^s_{33}}\big(b_{22}\big)$$

$$\times\alpha_{a^t_{65}a^s_{55}a^t_{54}a^s_{44}a^t_{43}a^s_{33}a^t_{32}a^s_{22}}\big(b_{11}\big)$$

Local surface transport X

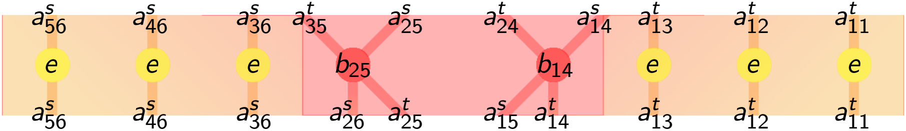


$$= \alpha_{a^s_{56}}(b_{45})\,\alpha_{a^s_{56}a^t_{55}a^s_{45}}(b_{34})\,\alpha_{a^s_{56}a^t_{55}a^s_{45}a^t_{44}a^s_{34}}(b_{23})\,\alpha_{a^s_{56}a^t_{55}a^s_{45}a^t_{44}a^s_{34}a^t_{33}a^s_{23}}(b_{12})$$

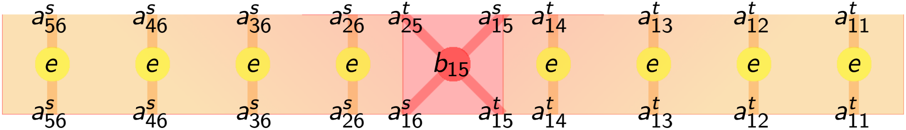


$$= \alpha_{a^s_{56}a^s_{46}}(b_{35})\,\alpha_{a^s_{56}a^s_{46}a^t_{45}a^s_{35}}(b_{24})\,\alpha_{a^s_{56}a^s_{46}a^t_{45}a^s_{35}a^t_{34}a^s_{24}}(b_{13})$$

Local surface transport XI



$$= \alpha_{a^s_{56}a^s_{46}a^s_{36}}\big(b_{25}\big)\alpha_{a^s_{56}a^s_{46}a^s_{36}a^t_{35}a^s_{25}}\big(b_{14}\big)$$



$$= \alpha_{a^s_{56}a^s_{46}a^s_{36}a^s_{26}}\big(b_{15}\big)$$

Local surface transport XII

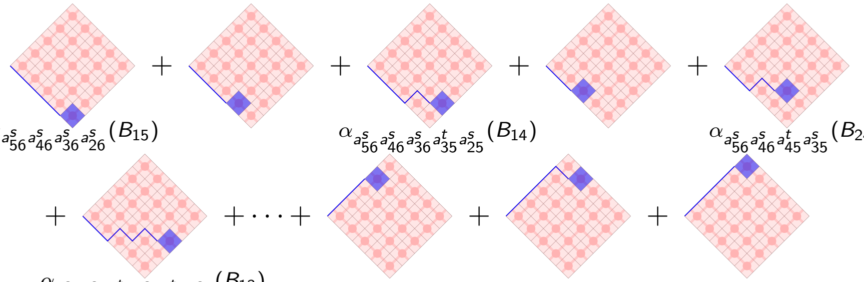
Composing vertically gives a rather big mess

$$
\begin{gathered}
\alpha_{a^t_{65}a^t_{64}a^t_{63}a^t_{62}}(b_{51}) \\
\alpha_{a^t_{65}a^t_{64}a^t_{63}}(b_{52})\alpha_{a^t_{65}a^t_{64}a^t_{63}a^t_{62}a^s_{52}}(b_{41}) \\
\alpha_{a^t_{65}a^t_{64}}(b_{53})\alpha_{a^t_{65}a^t_{64}a^t_{63}a^s_{53}}(b_{42})\alpha_{a^t_{65}a^t_{64}a^t_{63}a^s_{53}a^t_{52}a^s_{42}}(b_{31}) \\
\alpha_{a^t_{65}}(b_{54})\alpha_{a^t_{65}a^t_{64}a^s_{54}}(b_{43})\alpha_{a^t_{65}a^t_{64}a^s_{54}a^t_{53}a^s_{43}}(b_{32})\alpha_{a^t_{65}a^t_{64}a^s_{54}a^t_{53}a^s_{43}a^t_{42}a^s_{32}}(b_{21}) \\
b_{55}\alpha_{a^t_{65}a^s_{55}}(b_{44})\alpha_{a^t_{65}a^s_{55}a^t_{54}a^s_{44}}(b_{33})\alpha_{a^t_{65}a^s_{55}a^t_{54}a^s_{44}a^t_{43}a^s_{33}}(b_{22})\alpha_{a^t_{65}a^s_{55}a^t_{54}a^s_{44}a^t_{43}a^s_{33}a^t_{32}a^s_{22}}(b_{11}) \\
\alpha_{a^s_{56}}(b_{45})\alpha_{a^s_{56}a^t_{55}a^s_{45}}(b_{34})\alpha_{a^s_{56}a^t_{55}a^s_{45}a^t_{44}a^s_{34}}(b_{23})\alpha_{a^s_{56}a^t_{55}a^s_{45}a^t_{44}a^s_{34}a^t_{33}a^s_{23}}(b_{12}) \\
\alpha_{a^s_{56}a^s_{46}}(b_{35})\alpha_{a^s_{56}a^s_{46}a^t_{45}a^s_{35}}(b_{24})\alpha_{a^s_{56}a^s_{46}a^t_{45}a^s_{35}a^t_{34}a^s_{24}}(b_{13}) \\
\alpha_{a^s_{56}a^s_{46}a^s_{36}}(b_{25})\alpha_{a^s_{56}a^s_{46}a^s_{36}a^t_{35}a^s_{25}}(b_{14}) \\
\alpha_{a^s_{56}a^s_{46}a^s_{36}a^s_{26}}(b_{15})
\end{gathered}
$$

but we can visualize this mess by expanding out each b to lowest order (since we already know that the a's give the one-dimensional parallel transport, we don't have to expand them out).

Local surface transport XIII

The zeroth order term is just the identity. There are 25 terms with a single B (some of these terms are written underneath the pictures)



$\alpha_{a^s_{56} a^s_{46} a^s_{36} a^s_{26}}(B_{15}) + \cdots$

$\alpha_{a^s_{56} a^s_{46} a^s_{36} a^t_{35} a^s_{25}}(B_{14})$

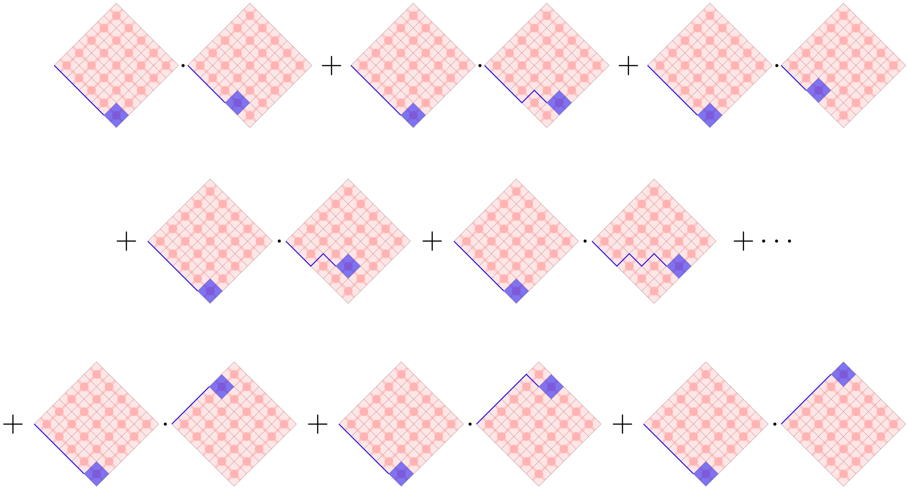
$\alpha_{a^s_{56} a^s_{46} a^t_{45} a^s_{35}}(B_{24})$

$\alpha_{a^s_{56} a^s_{46} a^t_{45} a^s_{35} a^t_{34} a^s_{24}}(B_{13})$

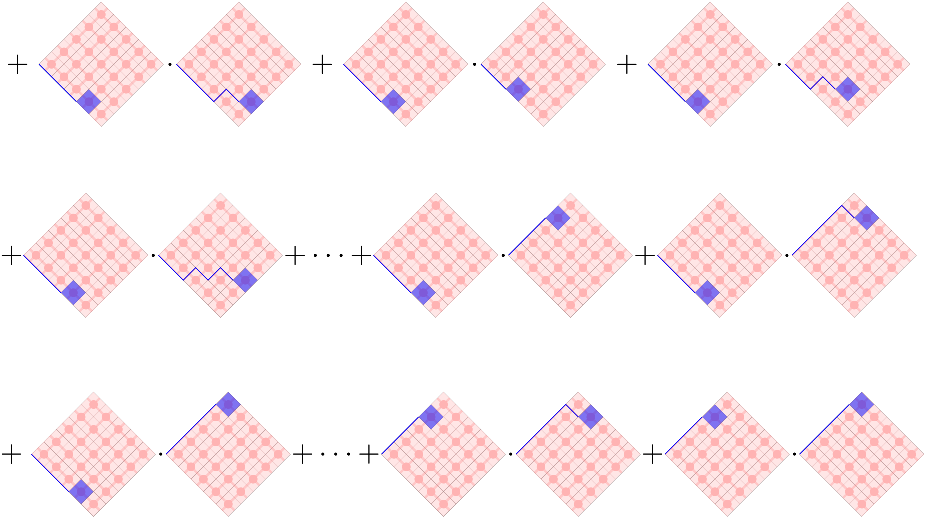
In other words, we calculate the ordinary parallel transport along a specified path between the point $(s, t) = (s_6, t_6)$ and another point (s_{i+1}, t_{j+1}) (represented by a blue line) and conjugate each B field at (s_i, t_j) (represented by a blue square) by that parallel transport. Then we sum over all points at which B has been specified.

Local surface transport XIV

There are $\sum_{k=1}^{24} k = \frac{24(25)}{2} = 600$, i.e. $\binom{25}{2}$, terms with two B's:



Local surface transport XV



Local surface transport XVI

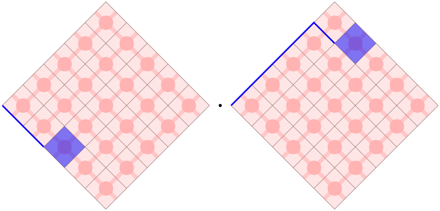
We should do this sum for all products of B's ranging from 0 to 25. The total number of all terms in such an expansion is enormous and is given by

$$\sum_{k=0}^{25} \binom{25}{k} = 2^{25},$$

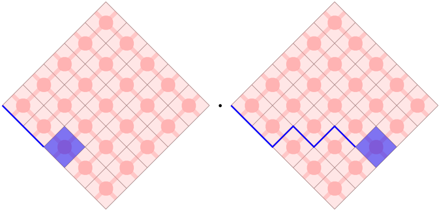
which is ridiculously huge (on the order of Avogadro's number). Fortunately, we can see a pattern by rearranging all of these terms.

Local surface transport XVII

For example, for terms with two B's, there are terms with two B's at different "heights" such as



and terms with B's at the same height such as



Local surface transport XVIII

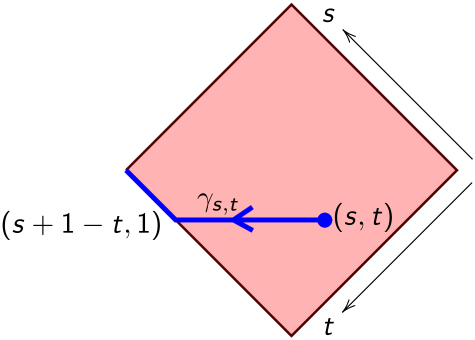
The ratio of terms with two B's at the same height to the total number of terms with two B's is ($n = 5$ in our picture)

$$\frac{\sum_{k=0}^{n} k(k-1)}{\binom{n^2}{2}} = \frac{2}{3n}.$$

Thus, as $n \to \infty$, the number of terms for which the B's are at the same height is a set of measure zero with respect to all possibilities. A similar argument applies for terms with k B's provided that $k \ll n$ (though I was too lazy to figure out the explicit formula). Since we will take $n \to \infty$ (as we normally do in calculus), this shows that we only care about terms at which B is inserted at different heights. This gives the following picture for the surface-iterated integral.

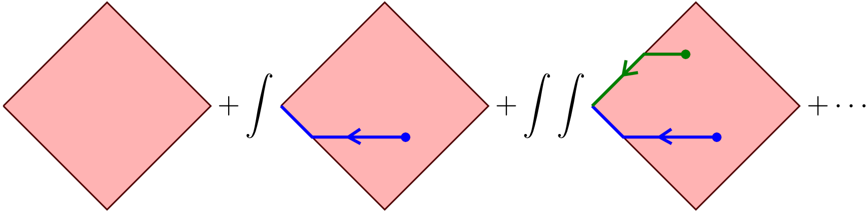
Local surface transport XIX

Let $\gamma_{s,t}$ be the path



Local surface transport XX

The limit of the result as $n \to \infty$ is given by an iterated integral



The surface-ordered integral is depicted schematically as an infinite sum of terms expressed by placing B at the endpoints of the drawn paths, conjugating it by parallel transport using A. Then we use an ordinary integral over the horizontal direction to get a 1-form. Finally we use the usual path-ordered integral in the vertical direction.

Special cases of local surface transport

- When $G = \{*\}$ and $H = U(1)$ the parallel transport is

$$\exp\left\{\int B_{\mu\nu} d\Sigma^{\mu\nu}\right\}$$

  and describes the coupling of a 2-form vector potential to the worldsheet of a charged string. In this case, B is known as a Kalb-Ramond field.
- When $H = G$, $\tau = \mathrm{id}_G$, and α is just conjugation, then this reproduces the non-abelian Stoke's theorem (see, for instance, work of Makeenko).
- When H is a covering space of G, this (technically a global version of this) can be used to calculate the flux of magnetic monopoles (see my paper).

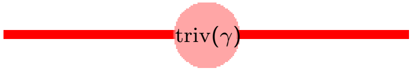
1-dimensional parallel transport is a functor I

One can think of (local) parallel transport of a path as a smooth functor

$$\mathcal{P}^1(M) \xrightarrow{\text{triv}} \mathbb{B}G$$

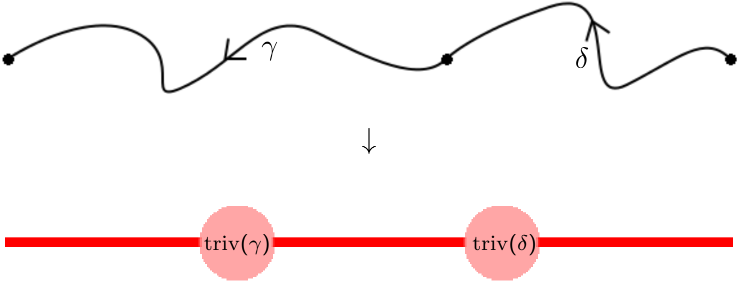
from the category of (thin) paths in a manifold M to the gauge group G viewed as a one-object category.

$\rightarrow$



1-dimensional parallel transport is a functor II

Functoriality just means path composition goes to group multiplication.



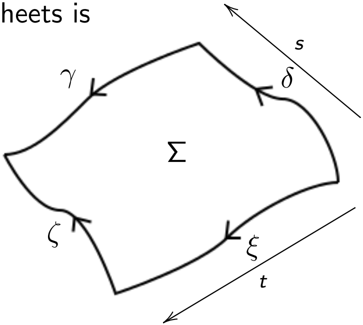
A result of Schreiber and Waldorf is that there is an equivalence between such functors and vector potentials $A \in \Omega^1(M; \mathfrak{g})$.

2-dimensional parallel transport is a functor I

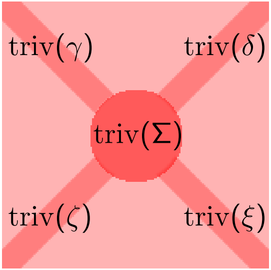
One can think of (local) parallel transport of a worldsheet as a smooth functor

$$\mathcal{P}^2(M) \xrightarrow{\text{triv}} \mathbb{B}(H, G, \tau, \alpha)$$

from the category of (thin) worldsheets in a manifold M to the crossed module (H, G, τ, α) viewed as a one-object 2-category. The assignment on strings and worldlines is the same as before. The assignment on worldsheets is

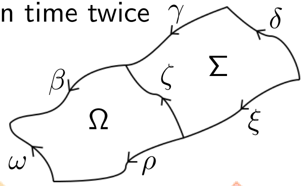


$\rightarrow$

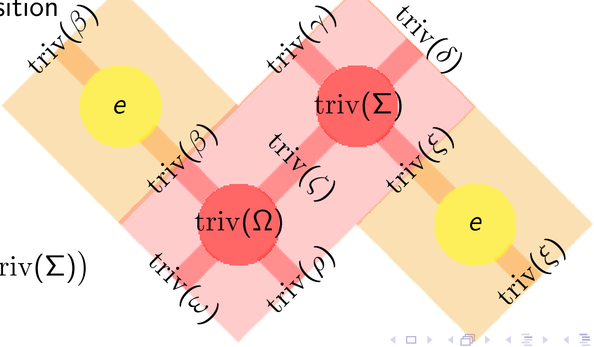


2-dimensional parallel transport is a functor II

In this case, functoriality means the previous condition for paths and in addition evolving a string in time twice



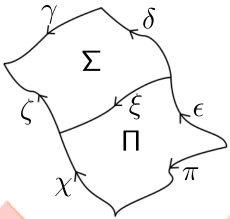
gives the composition



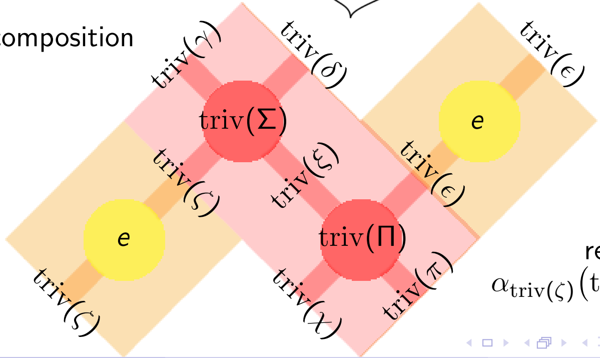
result:
$\mathrm{triv}(\Omega)\alpha_{\mathrm{triv}(\beta)}\big(\mathrm{triv}(\Sigma)\big)$

2-dimensional parallel transport is a functor III

while extending the string in space and evolving results in



gives the composition



result:

$$\alpha_{\mathrm{triv}(\zeta)}\big(\mathrm{triv}(\Pi)\big)\mathrm{triv}(\Sigma)$$

Gauge transformations for worldlines I

If A and A' are two different vector potentials related by an infinitesimal gauge transformation, then this gauge transformation can be described by a natural transformation of parallel transport functors

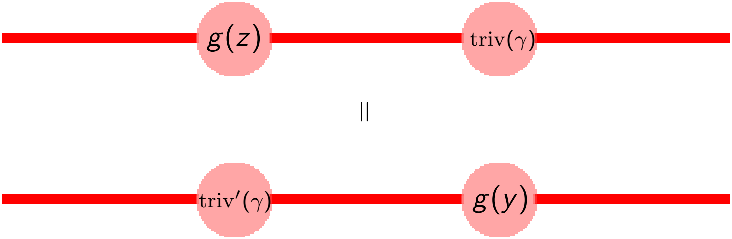
$$\begin{tikzcd} \mathcal{P}^1(M) \arrow[rr, bend left, "\mathrm{triv}"] \arrow[rr, bend right, "\mathrm{triv}'"'] \arrow[rr, phantom, "\Downarrow g"] & & \mathbb{B}G \end{tikzcd}$$

which is an assignment that sends a point $y \in M$ to a group element $g(y)$

Gauge transformations for worldlines II

satisfying the condition that for every path

the equality



holds.

First order gauge transformations for worldsheets I

Two pairs of gauge potentials (A, B) and (A', B') are gauge equivalent if there exists a pseudo-natural transformation of parallel transport functors

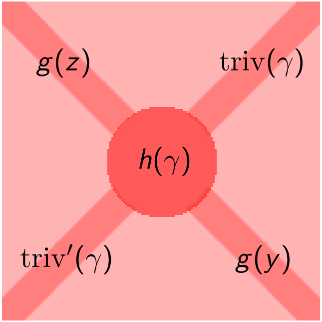
$$\begin{tikzcd} \mathcal{P}^2(M) \arrow[rr, bend left, "\mathrm{triv}"] \arrow[rr, bend right, "\mathrm{triv}'"'] \arrow[rr, phantom, "\Downarrow (g,h)"] & & \mathbb{B}G \end{tikzcd}$$

which is an assignment that sends a point $y \in M$ to a group element $g(y)$

First order gauge transformations for worldsheets II

and sends a path

to

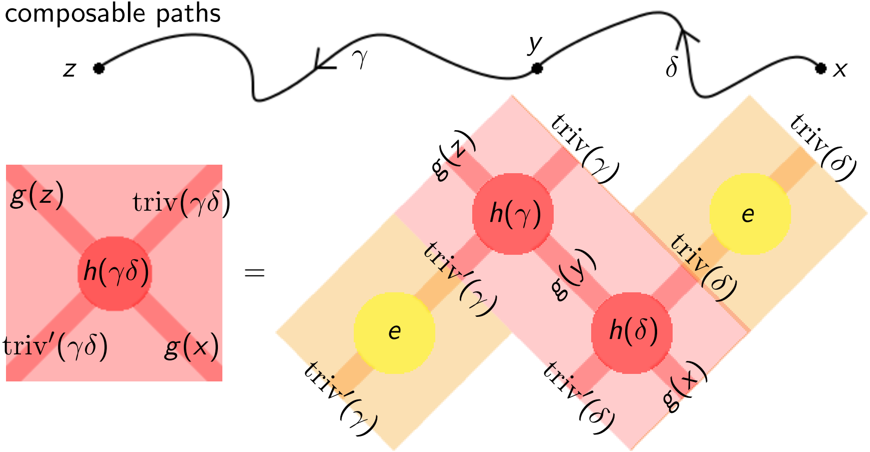


Note that, in particular, this says

$$\tau\big(h(\gamma)\big)\mathrm{triv}'(\gamma)g(y) = g(z)\mathrm{triv}(\gamma).$$

First order gauge transformations for worldsheets III

This assignment must satisfy two conditions. The first is that to a pair of composable paths

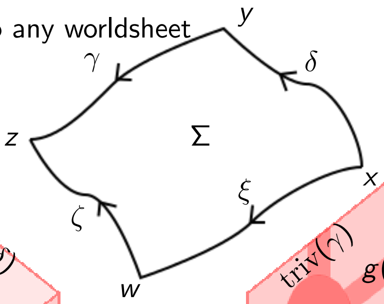


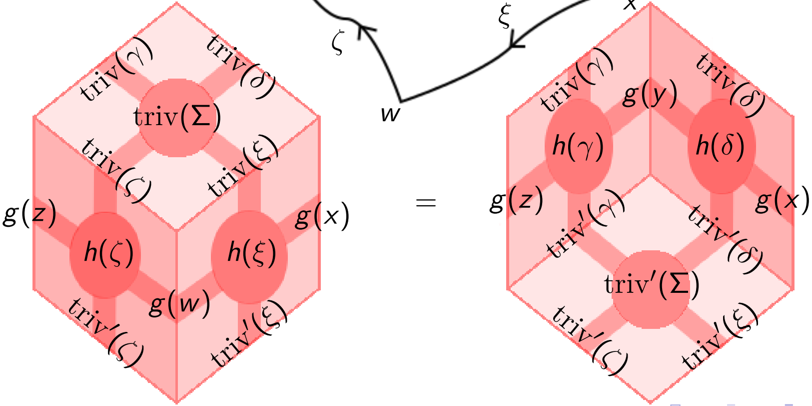
i.e.

$$h(\gamma\delta) = \alpha_{\mathrm{triv}'(\gamma)}\big(h(\delta)\big)\, h(\gamma).$$

First order gauge transformations for worldsheets IV

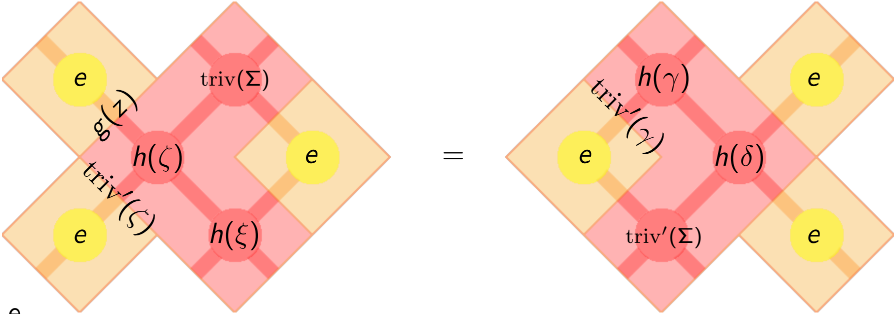
The second is that to any worldsheet





First order gauge transformations for worldsheets IV

where really we mean (by unfolding and placing in necessary identities and only the necessary 1-d defects have been labelled)



i.e.

$$\alpha_{\mathrm{triv}'(\zeta)}\big(h(\xi)\big)h(\zeta)\alpha_{g(z)}\big(\mathrm{triv}(\Sigma)\big) = \mathrm{triv}'(\Sigma)\alpha_{\mathrm{triv}'(\gamma)}\big(h(\delta)\big)h(\gamma)$$

or equivalently by our earlier condition

$$h(\zeta\xi)\alpha_{g(z)}\big(\mathrm{triv}(\Sigma)\big) = \mathrm{triv}'(\Sigma)h(\gamma\delta).$$

So much more to discuss!

- Second order gauge transformations
- Infinitesimal versions of gauge transformations
- Non-trivial 2-bundles and global parallel transport
- Gauge invariant observables (Wilson surfaces)
- Non-abelian magnetic monopoles
- etc...

Thank you + references

Thank you

Additional references:

- J. Baez and J. Huerta "An Invitation to Higher Gauge Theory" arXiv:1003.4485
- U. Schreiber and K. Waldorf "Connections on non-abelian Gerbes and their Holonomy" arXiv:0808.1923

Historical references:

- K. G. Wilson "Confinement of quarks" Phys. Rev. D. Vol. 10, No. 8, (1974) 2445–2459.
- M. Kalb and P. Ramond "Classical direct interstring action" Phys. Rev. D. Vol. 9, No. 8, (1974) 2273–2284
- C. Teitelboim "Gauge invariance for extended objects" Phys. Let. Vol. 167B, No. 1, (1985) 63–68

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