Two-dimensional algebra and gauge theory CCNY Student Research Symposium May 10, 2016

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Vector potential coupling to a charged particle I

A gauge field A couples to a charged particle of charge q, described by a trajectory $t \mapsto x(t)$, by a term in the Action of the form

$$q \int_x A \equiv q \int dt\, A_\mu(x(t)) \frac{dx^\mu}{dt}$$

(repeated indices are summed). Just to make things a bit more familiar, together with the kinetic term, the full Action would be

$$S[x] = m \int dt \sqrt{g\left(\frac{dx}{dt}, \frac{dx}{dt}\right)} + q \int dt\, A_\mu(x(t)) \frac{dx^\mu}{dt},$$

where g is the metric of spacetime. This produces the familiar Lorentz force law as the equations of motion (in the low velocity limit)

$$m \frac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \frac{d\vec{x}}{dt} \times \vec{B}\right).$$

Vector potential coupling to a charged particle II

Notice that the vector potential A does not appear in the equations of motion. However, the interaction term

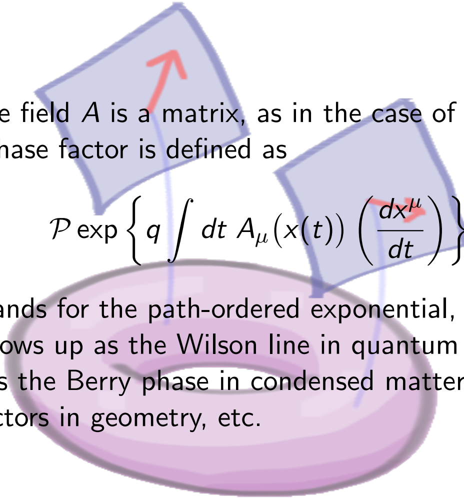
$$q \int_{\gamma} A$$

becomes really important in quantum mechanics. For example, it is responsible for phenomena such as the Aharanov-Bohm effect where it arises as the phase factor

$$\exp \left\{ q \int_{\gamma} A \right\}$$

and contributes to effects even when both the electric and magnetic fields vanish!

Non-abelian gauge fields



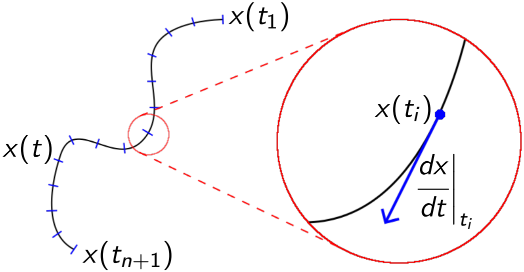
When the gauge field A is a matrix, as in the case of the strong force for example, this phase factor is defined as

$$\mathcal{P}\exp\left\{ q\int dt\; A_{\mu}(x(t))\left(\frac{dx^{\mu}}{dt}\right)\right\},$$

where $\mathcal{P}\exp$ stands for the path-ordered exponential, which we'll describe shortly. This shows up as the Wilson line in quantum chromodynamics in gauge theory, as the Berry phase in condensed matter, as the parallel transport of vectors in geometry, etc.

The path-ordered exponential I

The path-ordered exponential is constructed by breaking up a curve into infinitesimal paths



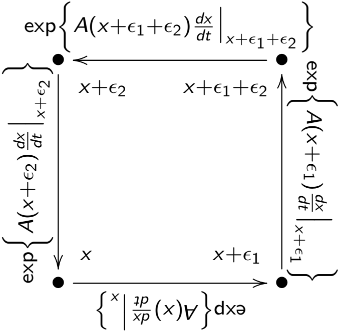
and associating the group elements

$$\exp\left\{ A_{\mu_i}(x(t_i)) \left.\frac{dx^{\mu_i}}{dt}\right|_{t_i} \right\}$$

at these paths and multiplying those group elements in the order dictated by the path.

The path-ordered exponential II

In the special case that the path is a tiny loop (say a square of side length ϵ), multiplying and expanding out to lowest order in ϵ gives



$$\simeq 1+\epsilon^2 F_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}+\mathcal{O}(\epsilon^3),$$

where F is the field strength. Notice that this kind of multiplication is *one-dimensional*.

The field strength

For example, in electromagnetism, F as a matrix is given by

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

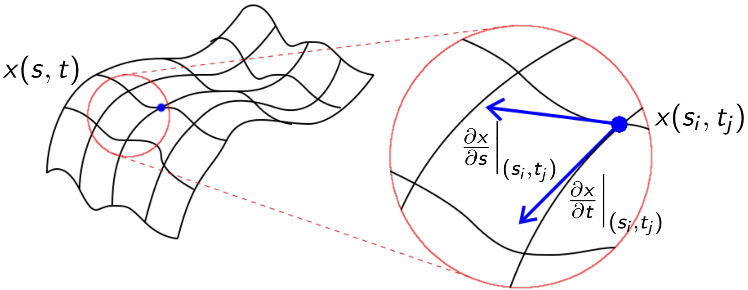
In 1974, Kenneth Wilson showed that exactly these infinitesimal squares can be used define an Action on a lattice that reproduces the term

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

which is the kinetic term for gauge bosons. He also showed the loops can be used to explain certain aspects of quark confinement.

2-dimensional surfaces

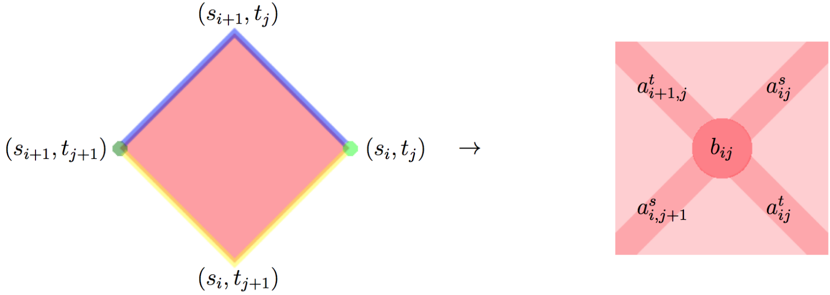
Similar questions can be asked for extended objects such as strings. Strings sweep out a two-dimensional surface in time.



We can break up the surface into infinitesimal pieces (they don't even have to be squares, but let's say they are).

2-form potential coupling to a charged string I

Taking one of these squares, we associate group elements from *two* different groups to faces *and* edges.



And yes, we do have to be careful about orientations. Flipping an orientation would correspond to taking an appropriate inverse of a group element.

2-form potential coupling to a charged string I

Here

$$a_{ij}^{s} := \exp\left\{ A_{\mu_i}\big(x(s_i,t_j)\big)\frac{\partial x^{\mu_i}}{\partial s}\bigg|_{(s_i,t_j)}\right\}, \; a_{ij}^{t} := \exp\left\{ A_{\nu_j}\big(x(s_i,t_j)\big)\frac{\partial x^{\nu_j}}{\partial t}\bigg|_{(s_i,t_j)}\right\}$$

denote the parallel transport along infinitesimal paths and

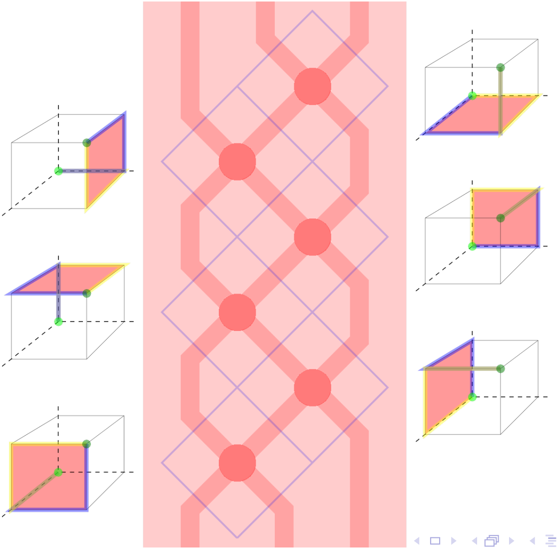
$$b_{ij} := \exp\left\{ -B_{\mu_i\nu_j}\big(x(s_i,t_j)\big)\frac{\partial x^{\mu_i}}{\partial s}\frac{\partial x^{\nu_j}}{\partial t}\bigg|_{(s_i,t_j)}\right\}$$

denotes the parallel transport along infinitesimal squares. B is the potential that couples to surfaces. Using category theory, one can use these basic ingredients to multiply in *two* dimensions!

Note: If we only associated group elements to faces, we would be *forced* to work with *abelian* gauge fields (that's a theorem).

A Wilson cube I

For example, along a cube the product looks like



A Wilson cube II

In the special case where the two groups used to label the faces and edges are the same, an infinitesimal cube gives

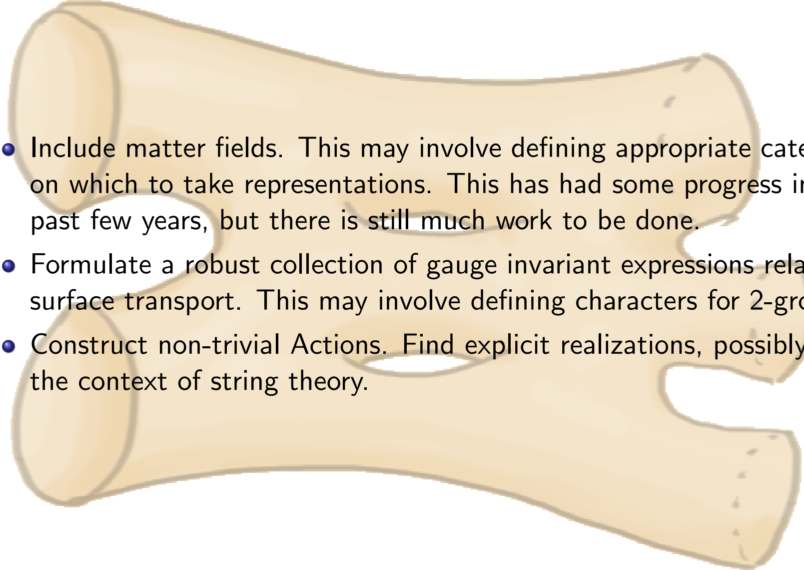
$$1 + \epsilon^3 H_{\mu\nu\lambda} \frac{\partial x^\mu}{\partial r} \frac{\partial x^\nu}{\partial s} \frac{\partial x^\lambda}{\partial t} + \mathcal{O}(\epsilon^4),$$

where

$$H_{\mu\nu\lambda} = \frac{\partial B_{\nu\lambda}}{\partial x^\mu} + \frac{\partial B_{\mu\nu}}{\partial x^\lambda} + \frac{\partial B_{\lambda\mu}}{\partial x^\nu} - [A_\mu, B_{\nu\lambda}] - [A_\nu, B_{\lambda\mu}] - [A_\lambda, B_{\mu\nu}]$$

is the analogue of the field strength for strings. Here $[\,\cdot\,,\,\cdot\,]$ is the commutator.

Future directions



- Include matter fields. This may involve defining appropriate categories on which to take representations. This has had some progress in the past few years, but there is still much work to be done.
- Formulate a robust collection of gauge invariant expressions related to surface transport. This may involve defining characters for 2-groups.
- Construct non-trivial Actions. Find explicit realizations, possibly in the context of string theory.

Thank you + references

Thanks for listening!

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Additional references (in my opinion, in increasing order of difficulty):

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