Notes on Gepner's talk at MSRI

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Abstract

These are some extra notes to Gepner's talk at MSRI. I do not necessarily write down everything in his talk. I try to draw some analogies to things I'm aware of to better understand the material (any errors I bring to the table are my own fault). I also include questions that I should answer at some point.

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1 Classical viewpoint on Thom spectra

Recall that an action of a topological group G on a space X is a continuous functor

$$\alpha : \mathcal{B}G \to \textbf{Spaces} \quad (1)$$
$$\bullet \mapsto X \quad (2)$$

from G viewed as a one-object topological category to some category of spaces. The quotient space X/G is the colimit of this functor. Since **Spaces** has a notion of homotopy, a *homotopy quotient* can be more useful. We denote this by $X//G$ or sometimes G^α (we will see why later). In particular, G acts on the point $*$ via the trivial action. The quotient space $*/G$ is just the point while the homotopy quotient is BG. The homotopy quotient is used to define the *equivariant cohomology* of a space S with a G action:

$$H^*_G(X) := H^*(X//G). \quad (3)$$

Gepner recalls that the Thom spectrum is something very similar to the homotopy quotient of an action of a group on a point. Instead, we consider a (suitably continuous) functor

$$\alpha : \mathcal{B}G \to \textbf{Spectra} \quad (4)$$
$$\bullet \mapsto \mathbb{S}, \quad (5)$$

where $\mathbb{S}$ is the sphere spectrum.

Question 1. What is the map α? I'm assuming it has a definition and it's not just some arbitrary map.

The sphere spectrum is the monoidal unit in **Spectra** just as the point is the monoidal unit in **Spaces**. The homotopy quotient $\mathbb{S}//G$ of the functor a is called the *Thom spectrum* and is denoted by MG. Therefore, one should think of BG and MG as similar objects but in different categories.

[From 3:30 to 7:00, Gepner gives several examples of Thom spectra]

2 Interlude on ∞-categories and topos theory

Since **Spaces** $\equiv \mathcal{S}$ is a model category, there is a natural ∞-category associated to it. We will therefore always consider $\mathcal{S}$ as an ∞-category of spaces.

Question 2. What is the construction that takes a model category and spits out an ∞-category?

Given a space (aka an ∞-groupoid) X, one can define the slice category $\mathcal{S}_{/X}$ over X of continuous maps (∞-functors) of spaces into X. Slice categories are examples of *presentable* ∞-categories. The ∞-category of presentable ∞-categories is denoted by **Pr**. It is a subcategory of ∞-**Cat**, the category of ∞-categories.

Question 3. What are the morphisms of this ∞-category? It seems like it will be a lot of data. Do we want the obvious diagram to commute, or for there to exist a homotopy (and therefore higher homotopies), or do we just want the diagram to commute up to homotopy?

Question 4. What are presentable ∞-categories?

Given a continuous map $f : Y \to X$, there is a pullback functor $f^* : \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ given by taking the (homotopy) pullback. Therefore, the slice construction defines a functor

$$\mathcal{S}_{/} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Pr} \tag{6}$$

$$X \mapsto \mathcal{S}_{/X}. \tag{7}$$

A given pullback associated to a map $f : Y \to X$ satisfies additional properties. Namely, there exists an adjoint triple

$$\mathcal{S}_{/X} \quad \begin{matrix} \xleftarrow{f_!} \\ \perp \\ \xrightarrow{f^*} \\ \perp \\ \xleftarrow{f_*} \end{matrix} \quad \mathcal{S}_{/Y} \ . \tag{8}$$

The notation mean that $f_!$ is left adjoint to f^* and f_* is right adjoint to f^*. Presentable ∞-categories whose spaces of morphisms that are both left (L) and right (R) adjoints form a subcategory of **Pr** denoted by $\mathbf{Pr}^{\mathrm{L,R}}$. Therefore, the slice construction defines a functor $\mathcal{S}_{/} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{L,R}} \to \infty\text{-}\mathbf{Cat}$.

Recall that the category of spaces has a Grothendieck topology on it.

Question 5. What is the definition of sites on ∞-categories? Furthermore, which topology is the one we're using on spaces? Are coverings local homeomorphisms?

The functor $\mathcal{S}_/$ satisfies several properties.

1. $\mathcal{S}_/$ is a sheaf (in the ∞ sense). People say that $\mathcal{S}_/$ satisfies *descent*. In particular, this implies that if $X = \mathrm{colim}_i X_i$, then $\mathcal{S}_{/X} \xrightarrow{\simeq} \lim_i \mathcal{S}_{/X_i}$, where again limits and colimits are in the ∞-categorical sense.

2. Given two spaces $Y \to X$ and $Z \to X$ in $\mathcal{S}_{/X}$, one can form the fiber product (this is just the homotopy pullback in $\mathcal{S}$). This product gives $\mathcal{S}_{/X}$ the structure of a symmetric monoidal ∞-category. Therefore, $\mathcal{S}_/$ factors through *commutative algebra objects* in $\mathbf{Pr}^{\mathrm{L,R}}$,
$$\mathcal{S}_/ : \mathcal{S}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}}). \tag{9}$$

   Question 6. Why do we say algebra? Don't we just have a commutative *monoid* object? What is the definition of a commutative *algebra* object?

   Remark 1. We can think of the category $\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$ as a category of nice symmetric monoidal model categories.

3. Because every space X is the colimit of its points
$$X \simeq \operatorname*{colim}_{* \to X} *, \tag{10}$$
   any sheaf (our example is $\mathcal{S}_/$) $\mathcal{S}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$ is determined by its value on the point $*$. We should therefore think of the sheaf over X as simply gluing several copies of this object over all points in a coherent fashion.

There is actually an equivalence between sheaves $\mathcal{S}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$ and commutative algebra objects in $\mathbf{Pr}^{\mathrm{L,R}}$. The reason for this is because spaces are ∞-categories. More explicitly, let $\mathcal{C}$ be such a commutative algebra object (note that in particular, it is an ∞-category). Then define the presheaf
$$\mathrm{Pre}_{\mathcal{C}}(\,\cdot\,) : \mathcal{S}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}}) \tag{11}$$
$$X \mapsto \mathrm{Pre}_{\mathcal{C}}(X) := \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C}). \tag{12}$$
It turns out this presheaf satisfies all the above properties. As a particular example, the ∞-category of spaces $\mathcal{S}$ itself is a commutative algebra object in $\mathbf{Pr}^{\mathrm{L,R}}$. It's a fact that the functors $\mathrm{Pre}_{\mathcal{S}}(\,\cdot\,)$ and $\mathcal{S}_/$ are naturally equivalent (this is one of the properties of an ∞-topos). To see this, for a space X, we have
$$\mathcal{S}_{/X} \simeq \mathcal{S}_{/\,\mathrm{colim}_{*\to X} *} \simeq \lim_{* \to X} \mathcal{S}_{/*} \simeq \lim_{* \to X} \mathcal{S} \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S}). \tag{13}$$
Think about that last equivalence more carefully. We have a category of spaces over every point in X (since $\mathcal{S}_{/*} \simeq \mathcal{S}$) and these spaces must all glue together nicely. This is precisely the right-hand-side.

This motivates the usage of the notation $\mathcal{C}_/ : \mathcal{S}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$ defined by $X \mapsto \mathcal{C}_{/X} := \mathrm{Pre}_{\mathcal{C}}(X)$ for any $\mathcal{C}$ in $\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$. This functor $\mathcal{C}_/$ satisfies many properties.

If $\mathcal{C}$ is stable and if $f : Y \to X$ is a proper map (which means that the homotopy fibers are compact), then f_* admits another right adjoint $f^!$
$$\mathcal{S}_{/X} \;\; \begin{matrix} \xleftarrow{f_!} \\ \perp \\ \xrightarrow{f^*} \\ \perp \\ \xleftarrow{f_*} \\ \perp \\ \xleftarrow{f^!} \end{matrix} \;\; \mathcal{S}_{/Y}\,. \tag{14}$$

Question 7. What does it mean for an ∞-category to be stable?

What are some examples of commutative algebra objects $\mathcal{C}$ in $\mathbf{Pr}^{\mathrm{L,R}}$? We will list some examples that will be useful for generalizing the notion of Thom spectra later.

Example 1. The ∞-category of spaces $\mathcal{S}$ was already mentioned above.

Example 2. The ∞-category of spectra Sp (the monoidal structure is the smash product).

Example 3. Let A be an A_∞, E_∞, or E_n ring spectrum (think associative or commutative ring). The ∞-category A-Mod of A-modules (spectra M equipped with a map $A \wedge M \to M$ satisfying the usual rules of modules in the ∞ sense) is an object of $\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$.

As a special case of the previous example, because the sphere spectrum is a unit in the category of spectra, $\mathbb{S}$-Mod is just Sp itself relating the third example to the second. So, the functor $\mathrm{Sp} : \mathcal{S}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$ sends a space X to $\mathrm{Pres}_{\mathrm{Sp}}(X)$, presheaves of spectra on X. This category $\mathrm{Pres}_{\mathrm{Sp}}(X)$ is actually monoidal. The monoidal product is pointwise smash product. The unit is

$$\mathbb{S}_X : x \mapsto \mathbb{S} \tag{15}$$

and sending all higher simplices to identities. This should be thought of as the trivial sphere spectrum over X. Equivalently, it is given by the pullback $p^*\mathbb{S}$ from the canonical map $p : X \to *$.

3 The Picard functor and invertible objects

Let $\mathbf{CAlg}^{\mathrm{gp}}(\mathcal{S})$ denote the ∞-category of group-like E_∞-spaces (∞-groupoids). There is a very special functor[1]

$$\mathrm{Pic} : \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}}) \to \mathbf{CAlg}^{\mathrm{gp}}(\mathcal{S}), \tag{16}$$

called the Picard functor, that takes a nice symmetric monoidal ∞-category $\mathcal{C}$ to $\mathrm{Pic}(\mathcal{C}) := \mathcal{C}^\times$, the *space* of *invertible* objects in $\mathcal{C}$. In particular, $\mathcal{C}^\times$ being a space implies that all morphisms are invertible (since paths and homotopies are invertible).

Example 4. Let $\mathcal{C} = \mathrm{Sp}$ be the category of spectra. Then

$$\mathrm{Pic}(\mathrm{Sp}) \simeq \mathrm{Pic}(\mathbb{S}) \simeq \mathbb{Z} \times BGL_1(\mathbb{S}). \tag{17}$$

Question 8. I do not understand this computation at all. What is going on? In particular, what is Picard applied to a *particular* spectrum? I thought it was a functor of ∞-categories.

Example 5. Let $\mathcal{C} = A$-Mod for A an A_∞, E_∞, or E_n ring spectrum. Then $\mathrm{Pic}(A\text{-Mod})$ is A-Line. Here A-Line refers to A-modules L that admit an equivalence $L \xrightarrow{\simeq} A$. This is because A-Line is the maximal ∞-groupoid in A-Mod generated by the A-lines (see Definition 3.11 [1]).

Example 6. Although I don't know how to compute this, I'm assuming the following is an example. Let X be a space. Then $\mathcal{S}_{/X}$ as discussed earlier is an object of $\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L,R}})$ and therefore $\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}})$. Thus, one should be able to make sense of $\mathrm{Pic}(\mathcal{S}_{/X})$.

[1]The L just means that morphisms are left adjoints.

Question 9. Does Pic preserve limits? Because then we can compute this by evaluating Pic on $\mathcal{S}$ itself. And if that is possible, then what is the subcategory of spaces of invertible objects? My guess is contractible spaces.

Remark 2. The purpose of the Picard functor in [1] is to look at the special case $\mathcal{C} = A$-Mod. Then one gets A-Line which is where coefficient modules live. A-Line is a classifying space for A-line bundles in some sense. We want to consider A-lines over a space X to specify the coefficients of a local cohomology theory. In other words, these varying A-lines will allows us to "twist" our cohomology theory, but I would prefer not to use the word "twist" since it's a bit over-used and simply say cohomology with coefficients in a line bundle. The totality of these A-lines will itself be an A-module, called the Thom spectrum. One can also think of it as the total space of some A-bundle in the generalized sense.

4 Modern viewpoint on Thom spectra

The Picard functor $\mathrm{Pic} : \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}}) \to \mathbf{CAlg}^{\mathrm{gp}}(\mathcal{S})$ satisfies an important property that is used in the generalization of Thom spectra.

Theorem 1 (ABG)**.** *Pic has a left adjoint and that adjoint is $\mathrm{Pre}_{\mathbf{CAlg}(\mathbf{Pr}^L)}$, presheaves, equipped with the Day convolution symmetric monoidal structure. This means there exist counit and unit natural transformations $\epsilon : \mathrm{Pre}_{\mathbf{CAlg}(\mathbf{Pr}^L)} \circ \mathrm{Pic} \Rightarrow \mathrm{id}_{\mathbf{CAlg}(\mathbf{Pr}^L)}$ and $\eta : \mathrm{id}_{\mathbf{CAlg}^{gp}(\mathcal{S})} \Rightarrow \mathrm{Pic} \circ \mathrm{Pre}_{\mathbf{CAlg}(\mathbf{Pr}^L)}$, respectively, satisfying the zig-zag identities.*

Since $\mathrm{Pic}(\mathcal{C})$ is a space, the counit defines a functor

$$\epsilon_{\mathcal{C}} : \mathrm{Pre}_{\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}})}(\mathrm{Pic}(\mathcal{C})) \equiv \mathrm{Fun}\left(\mathrm{Pic}(\mathcal{C})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}})\right) \simeq \mathcal{S}_{/\mathrm{Pic}(\mathcal{C})} \to \mathcal{C}. \tag{18}$$

Question 10. I do not see why the left-hand-side (the definition of the presheaf functor) is equivalent to $\mathcal{S}_{/\mathrm{Pic}(\mathcal{C})}$ yet Gepner writes the counit as having domain $\mathcal{S}_{/\mathrm{Pic}(\mathcal{C})}$.

Definition 1. The counit $\mathcal{S}_{/\mathrm{Pic}(\mathcal{C})} \to \mathcal{C}$ in the previous theorem is called the *generalized Thom spectrum functor*. It sends a functor $\alpha : X \to \mathrm{Pic}(\mathcal{C})$ to an object X^α of $\mathcal{C}$ called the *Thom spectrum* of $\alpha : X \to \mathrm{Pic}(\mathcal{C})$.

The reason for this terminology will be shown in Theorem 2. One thinks of X^α as the total space associated to $\alpha : X \to \mathrm{Pic}(\mathcal{C})$.

Proposition 1. *The counit from above is colimit preserving and symmetric monoidal.*

Now we can relate this discussion to the earlier one on Thom spectra.

Theorem 2. *Let $\alpha : \mathcal{B}G \to \mathrm{Sp}$ be a functor that sends $\bullet$ to $\mathbb{S}$ as at the beginning of this talk [I still don't know exactly what it is]. Now, this means we have a map at the level of spaces $\alpha : BG \to BGL_1(\mathbb{S})$ which is basically an object of $\mathcal{S}_{/\mathrm{Pic}(\mathrm{Sp})}$ since $\mathrm{Pic}(\mathrm{Sp}) \simeq \mathbb{Z} \times BGL_1(\mathbb{S})$ [we're ignoring the $\mathbb{Z}$ factor perhaps by assuming out spaces are connected for simplicity]. Under the counit from above, α (viewed as an object of $\mathcal{S}_{/\mathrm{Pic}(\mathrm{Sp})}$) gets sent to the Thom spectrum MG in the category* Sp.

We will now focus on $\mathcal{C} = A$-Mod with A a suitable ring spectrum as earlier discussed. Remember that in this case, $\mathrm{Pic}(A\text{-Mod})$ is A-Line.

Definition 2. Let $\alpha : X \to \mathrm{Pic}(A\text{-Mod}) = A\text{-Line}$ be an object of $\mathcal{S}_{/A\text{-Line}} \simeq \mathrm{Fun}(X^{\mathrm{op}}, A\text{-Line})$. Denote the associated Thom spectrum, the image of α under the counit map $\mathcal{S}_{/A\text{-Line}} \to A\text{-Mod}$, by X^α. The *A-twisted homology/cohomology of X with twist α* is defined as follows. The homology is

$$A_n^\alpha(X) := \pi_0 A\text{-Mod}(\Sigma^n A, X^\alpha) = \pi_n X^\alpha \tag{19}$$

and the cohomology is

$$A_\alpha^n(X) := \pi_0 A\text{-Mod}(X^\alpha, \Sigma^n A). \tag{20}$$

Remark 3. Now, why should this be called twisted homology and cohomology? In ordinary cohomology theory, we have a global coefficient ring A. This would correspond to a constant functor that sends every point $x \in X$ to the constant A-line A. For local coefficients, one simply has *some* A-line associated to every point of X. This is where the twist comes from and it is precisely represented by a map $\alpha : X \to \mathrm{Pic}(A\text{-Mod}) = A\text{-Line}$.

[From 47:30 until the end of the lecture, Gepner gives examples]

References

[1] Matthew Ando, Andrew J. Blumberg, and David Gepner, *Twists of K-theory and TMF*, Superstrings, Geometry, Topology, and C^*-algebras, 2010, pp. 27–63. ArXiv: 1002.3004.