Surface holonomy and monopoles (BUGCAT 2014)

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1. Ordinary parallel transport
   - Groups
   - Local transport
   - Global transport
2. Surface transport
   - 2-groups
   - Local surface transport
   - Global surface transport
   - Examples

Groups as categories

- Categories $\mathcal{C}$ have objects C_0 and morphisms C_1 along with functions

$$C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0, \qquad C_0 \xrightarrow{i} C_1$$

  specifying the source and target of a morphism along with an identity morphism for every object. There is also a composition map $C_1 \,{}_s\!\times_t C_1 \to C_1$ drawn as

$$z \xleftarrow{g_1} y \xleftarrow{g_2} x \quad \mapsto \quad z \xleftarrow{g_1 g_2} x$$

  that is associative and unital with respect to the identity morphism.
- Groups, G, can be thought of as categories, $\mathcal{B}G$, with one object all of whose morphisms are invertible.
- Lie groups can be described the same way with all sets and functions mentioned above being smooth.

Local parallel transport I

- Principal G-bundles $P \to M$ with connection can be described in terms of local $\mathfrak{g}$-valued 1-forms $(A_i \in \Omega^1(U_i; \mathfrak{g}))$ and transition functions $(g_{ij} \in \Omega^0(U_{ij}; G))$ for some open cover $\{U_i\}_{i \in I}$ of M.
- They give rise to, and are determined by, functors $\mathrm{tra} : \mathcal{P}_1(M) \to \mathcal{B}G$ with *local* (not global) smoothness properties. Here $\mathcal{P}_1(M)$ is a certain smooth groupoid whose points are paths and morphisms are certain equivalence classes of paths, known as *thin paths*. $\mathcal{P}_1(M)$ is called the *thin path groupoid* of M.
- One of the local smoothness conditions says that there are *smooth* functors $\mathrm{tra}_i : \mathcal{P}_1(U_i) \to \mathcal{B}G$ related to the restriction $\mathrm{tra}|_{U_i}$ by a smooth natural isomorphism. A functor is smooth when it is smooth on objects and morphisms. A natural transformation is smooth when it is a smooth function from objects to morphisms.

Local parallel transport II

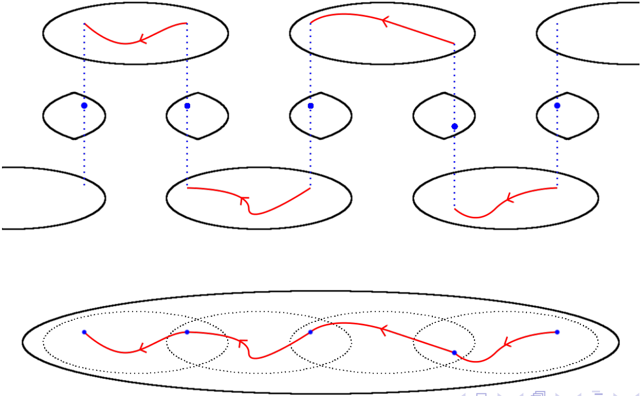
- For a path γ contained in a single open set U with associated local 1-form A, the smooth functor $\mathcal{P}_1(U) \to \mathcal{B}G$ is defined by the path-ordered integral

$$\mathcal{P}\exp\left\{\int_0^1 A_t\left(\frac{\partial}{\partial t}\right)dt\right\} := \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 dt_n\cdots\int_0^1 dt_1\;\mathcal{T}\left[A_{t_n}\left(\frac{\partial}{\partial t}\right)\cdots A_{t_1}\left(\frac{\partial}{\partial t}\right)\right],$$

  where $\mathcal{T}$ stands for time-ordering with earlier times appearing to the right.
- There is an isomorphism of categories between such differential forms A on U and such smooth functors as above.

Global parallel transport I

In general, we have to specify an open set for *every* point. Then, for paths γ not contained in a single open set, we have to *choose* a collection $(U_{i_n}, \ldots, U_{i_1})$ of open sets from $\{U_i\}$ that cover the path γ and choose points $(x_{n-1}, \ldots, x_1)$ on the intersections.



Global parallel transport II

- In this case, the transport is defined as

$$\mathrm{tra}(\gamma) = g_{i_{n+1} i_n}(x)\mathrm{tra}(\gamma_n) g_{i_n i_{n-1}}(x_{n-1}) \cdots \mathrm{tra}(\gamma_2) g_{i_2 i_1}(x_1) \mathrm{tra}(\gamma_1) g_{i_1 i_0}(x)$$

  where U_{i_0} is the open set chosen to contain x and $U_{i_{n+1}}$ is the open set chosen to contain y.

- This depends on our choice of open cover $\{U_i\}$ and our points in the intersections. For simplicity (and without loss of generality), assume we chose the same cover but we chose a new collection of open sets $(U_{i'_m}, \ldots, U_{i'_1})$ from $\{U_i\}$ and points $(x'_{m-1}, \ldots, x'_1)$. Then we define $\mathrm{tra}'(\gamma)$ analogously to the first case. The relationship between the two is

$$\mathrm{tra}'(\gamma) = g_{i'_m i_n}(y)\mathrm{tra}(\gamma) g_{i_1 i'_1}(x)$$

  where x and y are the beginning and endpoints of γ, respectively.

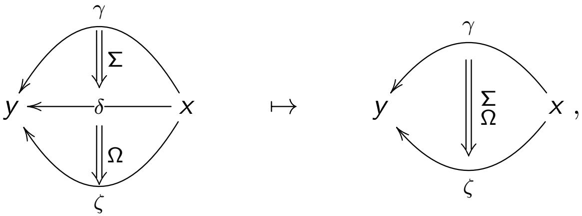
- Our next goal is to generalize these ideas and this relationship to parallel transport along surfaces. Then we'll give a simple example.

2-Groups are 2-categories I

- 2-Categories $\mathcal{C}$ have objects C_0, 1-morphisms C_1, and 2-morphisms C_2 along with functions

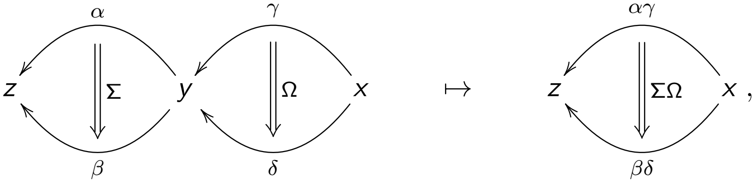
$$C_2 \;\begin{matrix}\xrightarrow{\;s\;}\\ \xleftarrow{\;i\;}\\ \xrightarrow[\;t\;]{}\end{matrix}\; C_1 \;\begin{matrix}\xrightarrow{\;s\;}\\ \xleftarrow{\;i\;}\\ \xrightarrow[\;t\;]{}\end{matrix}\; C_0$$

specifying the source and target of morphisms along with identity morphisms for every object and 1-morphism. There are also three composition maps $C_k \; {}_{s^{k-j}}\!\times_{t^{k-j}} C_k \to C_k$ for all j, k with $0 \leq j < k \leq 2$. The new compositions are drawn as



2-Groups are 2-categories II

and



and are both associative and unital with respect to the identity morphisms. Furthermore, they satisfy an important relation known as the interchange law.

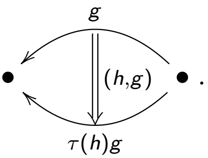
- 2-groups are 2-categories with one object and all 1- and 2-morphisms invertible with respect to all compositions.

2-Groups are 2-categories III

There is a one-to-one correspondence between 2-groups and *crossed modules*, i.e two groups $G(:= C_1)$ and $H(:= \ker s \subset C_2)$ together with group homomorphisms $\tau(:= t|_{\ker s}) : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$ (defined by $\alpha_g(h) := 1_g h 1_{g^{-1}}$) satisfying

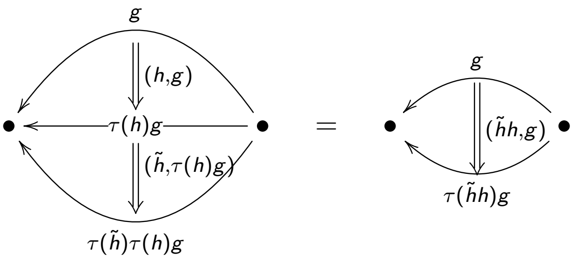
$$\alpha_{\tau(h)}(h') = hh'h^{-1} \quad \& \quad \tau(\alpha_g(h)) = g\tau(h)g^{-1}.$$

Using the above relationship, a 2-morphism looks like

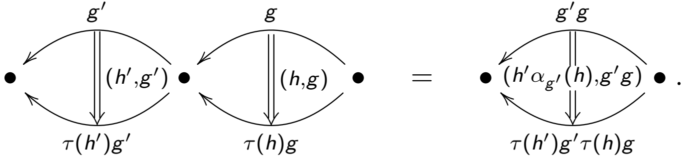


2-Groups are 2-categories IV

2-group compositions are written as



and



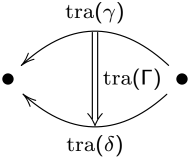
We will write the crossed module as $\mathcal{G}$ and the associated 2-group as $\mathcal{BG}$.

Local surface transport I

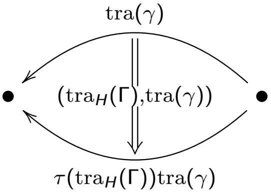
- Principal $\mathcal{G}$-2-bundles with connection are described in terms of local $\mathfrak{g}$-valued 1-forms $(A_i \in \Omega^1(U_i; \mathfrak{g}))$, $\mathfrak{h}$-valued 2-forms $(B_i \in \Omega^2(U_i; \mathfrak{h}))$, transition functions $(g_{ij} \in \Omega^0(U_{ij}; G))$, transition 1-forms $(\varphi_{ij} \in \Omega^1(U_{ij}; \mathfrak{h}))$, and higher transition functions $(f_{ijk} \in \Omega^0(U_{ijk}; H))$, for some open cover $\{U_i\}_{i\in I}$ of M.
- They give rise to, and are determined by, 2-functors $\mathrm{tra} : \mathcal{P}_2(M) \to \mathcal{BG}$ with *local* (not global) smoothness properties. Here $\mathcal{P}_2(M)$ is a certain "smooth" 2-groupoid whose points are paths, 1-morphisms are thin paths, and 2-morphisms are certain equivalence classes of bigons (homotopies), called thin bigons. $\mathcal{P}_2(M)$ is called the thin path 2-groupoid of M.
- Smoothness conditions are similar to the ordinary functor case.
- The formulas are more complicated, but there is also an equivalence of 2-categories between such differential data and smooth 2-functors as above.

Local surface transport II

Such a smooth 2-functor $\mathrm{tra} : \mathcal{P}_2(U) \to \mathcal{BG}$ assigns to bigons $\Gamma : \gamma \Rightarrow \delta$ an element



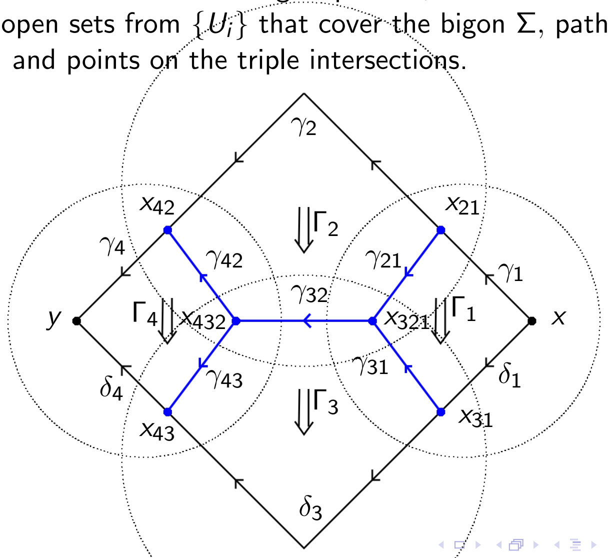
which we will also write as



to separate out the H and G components.

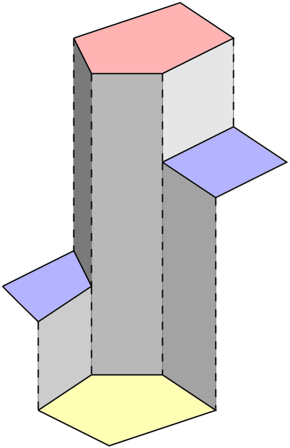
Global surface transport I

For bigons Σ not contained in a single open set, we have to choose a collection of open sets from $\{U_i\}$ that cover the bigon Σ, paths on the intersections, and points on the triple intersections.



Global surface transport II

Even in this relatively simple case, the transport is very complicated and is given by the diagram

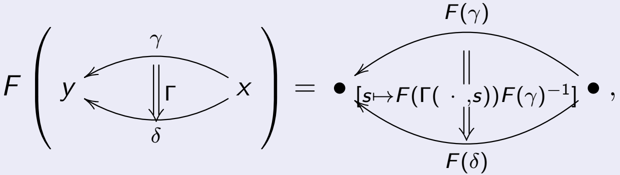


Covering 2-groups

One example of a 2-group is the universal covering 2-group where $\tau : H \to G$ is a universal cover with $H = P_e G/\sim$ (homotopy classes of paths starting at $e \in G$) and α is conjugation. Multiplication in H is by multiplying representatives. From now on, $\mathcal{G}$ will signify such a 2-group.

Theorem (P)

The formula for local parallel transport for any bigon under any smooth 2-functor $F : \mathcal{P}_2(X) \to \mathcal{BG}$ is given by the formula



where $[s \mapsto g(s)]$ is the homotopy class of a path in G starting at $e \in G$ and is therefore an element of H. In particular, it is determined by the value on paths and the homotopy class of bigons.

$SU(2)$ monopoles I

A simple example of a universal covering 2-group $\tau : SU(2) \to SO(3)$. Let $P \to S^2$ be a nontrivial principal $SO(3)$ bundle over S^2 with connection given by

$$A_N := \frac{J_3}{2}(1 - \cos\theta)d\phi \qquad \& \qquad A_S := -\frac{J_3}{2}(1 + \cos\theta)d\phi$$

and transition function

$$g_{NS}(\phi) := e^{-\phi J_3}$$

after choosing the cover to be the northern and southern hemispheres. Here

$$J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \& \quad J_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the generators of $\mathfrak{so}(3)$.

$SU(2)$ monopoles II

We define a principal $\mathcal{G}$-2-bundle by setting

$$B_N := \frac{\sigma_3}{4i} \sin\theta \; d\theta \wedge d\phi \quad \& \quad B_S := \frac{\sigma_3}{4i} \sin\theta \; d\theta \wedge d\phi,$$

where

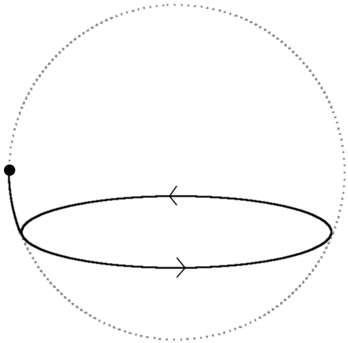
$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \& \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

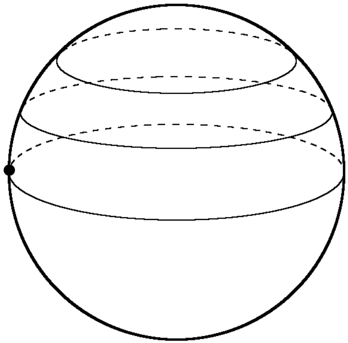
are the Pauli matrices, a set of generators for $\mathfrak{su}(2)$. We demand that the transition 1-forms vanish. All other data have been specified above. For reference, the map $\underline{\tau} : \mathfrak{su}(2) \to \mathfrak{so}(3)$ induced from the covering map on Lie algebras is given by

$$\underline{\tau}\left(\frac{1}{2i}\sigma_i\right) = J_i.$$

$SU(2)$ monopoles III

We conveniently choose the following one-parameter family of paths to describe the sphere as a bigon.

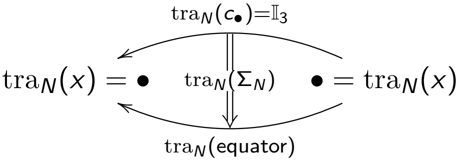




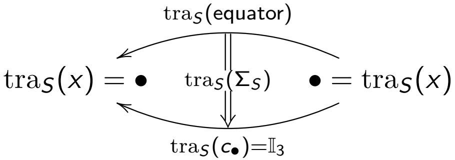
With these paths, there is no contribution from the path-ordered integral when traversing the θ direction.

$SU(2)$ monopoles IV

For the northern hemisphere, we have



while for the southern hemisphere, we have



$SU(2)$ monopoles V

To paste these two bigons together vertically, we have to apply the gauge transformations. First, since x is at $\phi = 0$, $g_{NS}(x) = 1$. Furthermore, since $\varphi_{NS} = 0$, the gauge transformation taking us from the northern hemisphere to the southern hemisphere is trivial (it is the identity). This is reasonable because

$$\mathrm{tra}_N(\text{equator}) = \mathrm{tra}_S(\text{equator})$$

since the lefthand side is

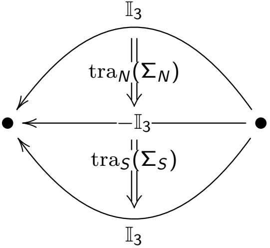
$$\mathrm{tra}_N(\text{equator}) = \mathcal{P}\exp\left\{\int_0^{2\pi} \frac{J_3}{2}\left(1 - \cos\left(\frac{\pi}{2}\right)\right) d\phi\right\} = e^{J_3\pi} = -\mathbb{I}_3$$

while the righthand side is

$$\mathrm{tra}_S(\text{equator}) = \mathcal{P}\exp\left\{-\int_0^{2\pi} \frac{J_3}{2}\left(1 + \cos\left(\frac{\pi}{2}\right)\right) d\phi\right\} = e^{-J_3\pi} = -\mathbb{I}_3.$$

$SU(2)$ monopoles VI

Therefore we can vertically compose both of the 2-morphisms and this gives the bigon describing the entire sphere as a bigon from the constant loop to itself



which indeed is a homotopy class of a loop based at the identity in $SO(3)$. The calculation of this loop has been known for a long time, but the above analysis gives a rigorous justification for its composition in terms of surface transport of principal 2-bundles. Nevertheless, we compute it.

$SU(2)$ monopoles VII

In the ϕ direction, the potential only depends on J_3 and can be computed as an ordinary integral. For the bigon describing the northern hemisphere, the transport is given by

$$\begin{aligned}\mathrm{tra}_N(\Sigma_N) &= \left[\theta \mapsto e^{\int_{\Sigma_N(\,\cdot\,,\theta)} A_N}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\theta \mapsto e^{\frac{J_3}{2}\int_0^{2\pi}(1-\cos\theta)d\phi}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\theta \mapsto e^{\pi J_3(1-\cos\theta)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}.\end{aligned}$$

Similarly, the southern hemisphere gives

$$\begin{aligned}\mathrm{tra}_S(\Sigma_S) &= \left[\theta \mapsto e^{\int_{\Sigma_S(\,\cdot\,,\theta)} A_S}(-\mathbb{I}_3)\right]_{\theta=\frac{\pi}{2}}^{\theta=\pi} \\ &= \left[\theta \mapsto -e^{-\pi J_3(1+\cos\theta)}\right]_{\theta=\frac{\pi}{2}}^{\theta=\pi}.\end{aligned}$$

$SU(2)$ monopoles VIII

The compose these vertically, which means multiplying the representatives pointwise and taking the homotopy class, we need to make the paths have the same domain. Thus by shifting θ by $\frac{\pi}{2}$ in $\mathrm{tra}(\Sigma_S)$, we can write it as

$$\begin{aligned}\mathrm{tra}_S(\Sigma_S) &= \left[\theta \mapsto -e^{-\pi J_3\left(1+\cos\left(\theta+\frac{\pi}{2}\right)\right)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\theta \mapsto -e^{-\pi J_3(1-\sin\theta)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} .\end{aligned}$$

Finally, composing gives

$$\mathrm{tra}_S(\Sigma_S)\mathrm{tra}_N(\Sigma_N) = \left[\theta \mapsto -e^{\pi J_3(\sin\theta-\cos\theta)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} .$$

As θ runs from 0 to $\frac{\pi}{2}$, $\sin\theta - \cos\theta$ runs from -1 to 1 monotonically. Thus, we get a loop at the identity completing exactly one full rotation along the z axis. Therefore, this loop is given by the element $-\mathbb{I}_2$, the nontrivial element in $SU(2)$ over the identity element in $SO(3)$.