Bendersky on elliptical genera and the Witten genus

Notes by Arthur Parzygnat

October 8, 2014

Let M be a $4k$-dimensional manifold. Recall what the signature of M is. We have a cup product $H^{2k} \otimes H^{2k} \to H^{4k} \cong \mathbb{Z}$. Now $H^{2k} = \langle \alpha_1, \ldots, \alpha_r \rangle$. We get a matrix from this whose matrix components are given by

$$a_{ij} = \langle \alpha_i \cup \alpha_j, [M] \rangle. \tag{1}$$

We can diagonalize it. When we do, we get a number p of positive eigenvalues and a number n of negative eigenvalues. Then $p - n =: \mathrm{Sign}(M)$. This is the signature of M. The signature has a nice property, which I wasn't able to write down because Bendersky erased it. It has to do with fibrations.

Definition 1. A genus is a ring homomorphism $\varphi : \Omega_*^{SO} \otimes \mathbb{Q} \to R$ where R is both a ring and a $\mathbb{Q}$-module.

Definition 2. Ω_*^{SO} is the oriented cobordism ring defined as follows. For every manifold M we embed it in some large enough Euclidean space, which corresponds to some classifying map $M \to BO$ by taking the normal bundle. We assume that we can lift this map to $M \to BSO$. The claim is that this map does not depend on the embedding *up to homotopy*.

The reason we use normal bundles instead of tangent bundles is because $\Omega_*^{SO} = \pi_0(MSO)$.

Example 1. The signature $\varphi := \mathrm{Sign}$ is an example of a genus.

The motivating question is: given a fibration $F \to E \to M$ where E and M are simply connected, which genera φ satisfy $\varphi(E) = \varphi(F)\varphi(M)$?

Before answering this, first we claim that

$$\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]. \tag{2}$$

This fact is described in Milnor-Stasheff. It says that any $4k$-dimensional manifold M is (rationally) a product of different $\mathbb{CP}^{2n}$ provided that the dimensions make sense. This also says that when M is not $4k$-dimensional, then it is cobordant (rationally) to the empty manifold.

Definition 3. We define a power series in the variable x in the $\mathbb{Q}$-module R for any genus φ

$$g(x) := \sum_{n=0}^{\infty} \frac{\varphi(\mathbb{CP}^{2n})}{2n+1} x^{2n+1}. \tag{3}$$

It turns out that

$$g(x) = \int_0^x \sum_{n=0}^{\infty} \varphi(\mathbb{CP}^{2n}) t^{2n} dt \tag{4}$$

We say that φ is *elliptic* if

$$g(x) = \int_0^x (1 - 2\delta t^2 + \epsilon t^4)^{-1/2} dt \tag{5}$$

for some $\delta, \epsilon \in R$, where R is the $\mathbb{Q}$-module.

Example 2. The power series for the signature is

$$g(x) = \int_0^x \frac{1}{\sqrt{1 - t^2 + t^4}} dt. \tag{6}$$

The motivation for the above definition is the following theorem due to Serge Ochanine.

Theorem 1. *φ is multiplicative, i.e. $\varphi(E) = \varphi(F)\varphi(M)$, for a fibration $F \to E \to M$ of 1-connected manifolds* if and only if *φ is elliptic.*

How does one recover the genus from the function $g(x)$?

Consider a power series algebra $R[[x_1, \ldots, x_l]]$. We have the elementary symmetric functions $\sigma_i(x_1, \ldots, x_l)$ that satisfy

$$\prod_{i=1}^{l} (1 + x_i t) = \sum_{i=1} \sigma_i t^i. \tag{7}$$

For example,

$$\sigma_1 = x_1 + \cdots + x_l. \tag{8}$$

Let $R[[x_1, \ldots, x_l]]^{\Sigma_l}$ be the power series that are invariant under Σ_l. This is a subset of the polynomial algebra. Then

$$R[[x_1, \ldots, x_l]]^{\Sigma_l} = R[[\sigma_1, \ldots, \sigma_l]]. \tag{9}$$

The Pontryagin classes

$$P_i(M) \in H^{4i}(M; \mathbb{Z}) \tag{10}$$

are defined by

$$P_i(M) := c_{2i}(TM \otimes \mathbb{C}), \tag{11}$$

the even Chern-classes of the complexified tangent bundles. We can generalize this to any bundle $E \to M$ as well.

Then we can write the total Pontryagin class as

$$P(M) := \sum_{i=0} P_i(M). \tag{12}$$

We can formally write this as

$$P(M) = \prod_{i=0} (1 + y_i^2) \tag{13}$$

where $|y_i| = 2i$ is the degree and the y's are the symmetric generators. In other words, the Pontryagin classes are the symmetric functions on the y_i^2. For our purposes, we don't need to know the exact form of the y_i^2. Now, given a $4k$-dimensional manifold M, we can define a Pontryagin number associated to any sequence of natural numbers $I = (i_1, \ldots, i_t)$ satisfying

$$\sum_{j=1}^{t} i_j = k \tag{14}$$

by

$$\langle P_I \rangle := \langle P_{i_1} \cdots P_{i_t}, [M] \rangle \in \mathbb{Z}. \tag{15}$$

The claim is that

$$\langle P_I \rangle = 0 \tag{16}$$

if $M = 0$ in Ω_*^{SO}. However, $\langle P_I \rangle$ is not a genus (with values in $\mathbb{Z}$)!

Let $f(x)$ be an arbitrary power series. Then we can define

$$\prod_{i=1}^{l} f(y_i^2) = F(P_1, \ldots) \tag{17}$$

a power series in the Pontryagin classes.

Definition 4. Let $s(u)$ be the power series in u that is the formal inverse of $g(x)$. This means that

$$s(g(u)) = u, \tag{18}$$

which means

$$\int_0^{s(u)} \sum \varphi(\mathbb{CP}^{2n}) t^{2n} dt = u. \tag{19}$$

Theorem 2. *In the notation from above,*

$$\varphi(M) = \left\langle \prod_i \frac{y_i}{s(y_i)}, [M] \right\rangle \tag{20}$$

Example 3. This is the Hirzebruch L formula when $\varphi = \text{Sign}$.

Proposition 1. *Regarding the formula*

$$g(x) = \int_0^x (1 - 2\delta t^2 + \epsilon t^4)^{-1/2} dt \tag{21}$$

we have the following facts.

(i) $\delta = \varphi(\mathbb{CP}^2)$, *where* $|\delta| = 2$.

(ii) $\epsilon^k = \varphi(\mathbb{HP}^{2k})$, *where* $|\epsilon| = 4$.

(iii) $\varphi(\Omega_*^{SO}) = \mathbb{Z}[\delta, 2\gamma, 2\gamma^2, \ldots, 2\gamma^{2^t}, \ldots]$, *where* $\gamma = \frac{(\delta^2 - \epsilon)}{4}$.

More on this is in Stong's book. Item (iii) follows from the Atiyah-Singer Index theorem.

Now, $\mathbb{HP}^{2k}$ is cobordant to some polynomial of even complex projective spaces and these are determined by their Pontryagin numbers (rationally). But let's not complicate things by writing down the formula.

Definition 5. The category of genera is given by the following. An object is a genus $\varphi : \Omega_*^{SO} \to R$. A morphism from $\varphi : \Omega_*^{SO} \to R$ to $\varphi : \Omega_*^{SO} \to S$ is a ring morphism from $R \to S$ that makes the diagram commute.

Spoiler alert:

Theorem 3. *The Witten genus is an initial object in this category.*

Let E be a finite-dimensional complex vector bundle. We define the following power series

$$S_t(E) := 1 + S^1(E)t + \cdots S^i(E)t^i + \cdots \tag{22}$$

and

$$\Lambda_t(E) := 1 + \Lambda^1(E)t + \cdots \Lambda^i(E)t^i + \cdots , \tag{23}$$

where S stands for symmetric product and Λ is the anti-symmetric (exterior product). We also define the even power series

$$S_t^{\text{ev}} := \bigotimes_{n\geq 1} S_{t^{2n}}(E) \tag{24}$$

and

$$\Lambda_t^{\text{ev}} := \bigotimes_{n\geq 1} \Lambda_{t^{2n}}(E), \tag{25}$$

where the product is the formal tensor product of power series. We also define

$$B_q(E) := S_q^{\text{ev}}(E) \otimes \Lambda_q^{\text{ev}}(E) = 1 + (2E)q^2 + \left(\Lambda^2(E) \oplus S^2(E) \oplus (2E) \oplus E^{\otimes 2}\right)q^4 + \cdots \tag{26}$$

Recall the following Cartan formula

$$\Lambda^k(E \oplus F) = \bigoplus \Lambda^i(E) \wedge \Lambda^{k-i}(F) \tag{27}$$

from which it follows that

$$\Lambda_t(E \oplus F) = \Lambda_t(E) \wedge \Lambda_t(F). \tag{28}$$

Then set

$$\Lambda_t(0) = 1. \tag{29}$$

When we have this, we can extend our definition of B_q to virtual vector bundles. Then we have

$$\Lambda(E \oplus (-E)) = 1 \tag{30}$$

and

$$\Lambda(-E) = \frac{1}{\Lambda(E)}. \tag{31}$$

Definition 6. Define

$$\Phi(M) := \langle 2^{2k}\hat{L}(M)\text{ch}(B_q(TM \otimes \mathbb{C} - 4k\mathbb{C})), [M]\rangle \in \mathbb{Q}[[q]] \tag{32}$$

where $\hat{L}$ is the genus associated to the power series

$$g(x) = \frac{x}{2\tanh\left(\frac{x}{2}\right)}. \tag{33}$$

The R for the Witten genus is $\mathbb{Q}[[q]]$. Here $(TM \otimes \mathbb{C} - 4k\mathbb{C})$ is a bundle, where we think of $\mathbb{C}$ as the trivial bundle.

Where does this come from?

Hirzebruch asked if we can find some genus L so that $L(M)$ picks out a Pontryagin number and gives 1. The power series written above is what he came up with.

Witten proved the following.

Theorem 4. Φ *defined above is an elliptic genus.*

Proof. The δ and ϵ are defined by

$$\delta = \frac{1}{2}\prod_{n\geq 1}(1-q^{2n})^4\left\{\prod_{m\geq 1}(1+q^{2m+1})^8 + \prod_{m\geq 1}(1-q^{2m+1})^8\right\} \tag{34}$$

$$= 1 + 24q^2 + 24q^4 + \cdots \tag{35}$$

and

$$\epsilon = \prod_{n\geq 1}(1-q^{2n})^8(1+q^{2n-1})^8(1-q^{2n-1})^8 \tag{36}$$

$$= 1 - 16q^2 + 112q^4 + \cdots \tag{37}$$

□

Remark 1. This is related to modular forms.

We will now define a cohomology theory. This is called *elliptic cohomology.*

Definition 7. $\Omega_*^{SO}(X) :=$ elements M of Ω_*^{SO} together with a map $f : M \to X$ modded out by an equivalence relation. We write elements as pairs (M, f). We say that $(M, f) = 0$ if $M = \partial W$ and f extends to $F : W \to X$. This defines the equivalence relation.

Theorem 5. *The assignment*

$$X \mapsto \Omega_*^{SO}(X) \otimes_\Phi \mathbb{Z}[\epsilon, \delta] \tag{38}$$

defines a homology theory. Note that this tensor product means that for a pair

$$M \cdot (N, f) \otimes p \equiv (N, f) \otimes \Phi(M) \cdot p, \tag{39}$$

which makes sense because $M \in \Omega_*^{SO}$ *acts on the left on this module since* $\Omega_*^{SO} \equiv \Omega_*^{SO}(\bullet)$ *acts on* $(N, f) \in \Omega_*^{SO}(X)$ *and* $\Phi(M)$ *acts on* $p \in \mathbb{Z}[\epsilon, \delta]$.