Kruskel-Szekeres coordinates and Penrose diagrams

Arthur J. Parzygnat

December 21, 2014

Abstract

The following notes are a variation of `http://www.physik.uniregensburg.de/forschung/wegscheider/gebhardt_files/skripten/Lect04KruskalCoord.pdf` and have the necessary corrections. I have also used chapter 7 of Sean Carroll's lecture notes on general relativity available at `http://preposterousuniverse.com/grnotes/`. Mathematica was used for all plots. We set the speed of light $c = 1$ throughout. Absolutely no originality is claimed here.

Kruskel-Szekeres coordinates

The Schwarzschild metric is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{1}$$

where r_s is the horizon radius

$$r_s := 2GM, \tag{2}$$

where M is the mass of the black hole. We will ignore the angular coordinates and focus only on the time and radial components. Note that the metric is well-defined only on the patch

$$(t, r) \in \mathbb{R} \times (r_s, \infty). \tag{3}$$

The goal is to find a new coordinate patch that extends beyond the horizon. This is the point of Kruskel-Szekeres coordinates. The (t, r) coordinates are what an asymptotic observer might naturally use. However, an infalling observer might use different coordinates. Physics does not depend on the choice of coordinates.

Rather than going through the step-by-step derivation of these Kruskel-Szekeres coordinates, we define them explicitly here. Let T and R be a new coordinate system defined on a patch

$$(T, X) \in \mathbb{R} \times \mathbb{R} \tag{4}$$

in terms of $(t, r) \in \mathbb{R} \times (r_s, \infty)$ by

$$T := \left(\frac{r}{r_s} - 1\right)^{1/2} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right) \tag{5}$$

$$X := \left(\frac{r}{r_s} - 1\right)^{1/2} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right). \tag{6}$$

Notice that this only defines (T, X) in the domain

$$(T, X) \in \mathbb{R} \times (0, \infty). \tag{7}$$

However, notice that for $r \in (0, r_s)$, the coordinates (T, R) do not make sense. However, by a redefinition, one can also define

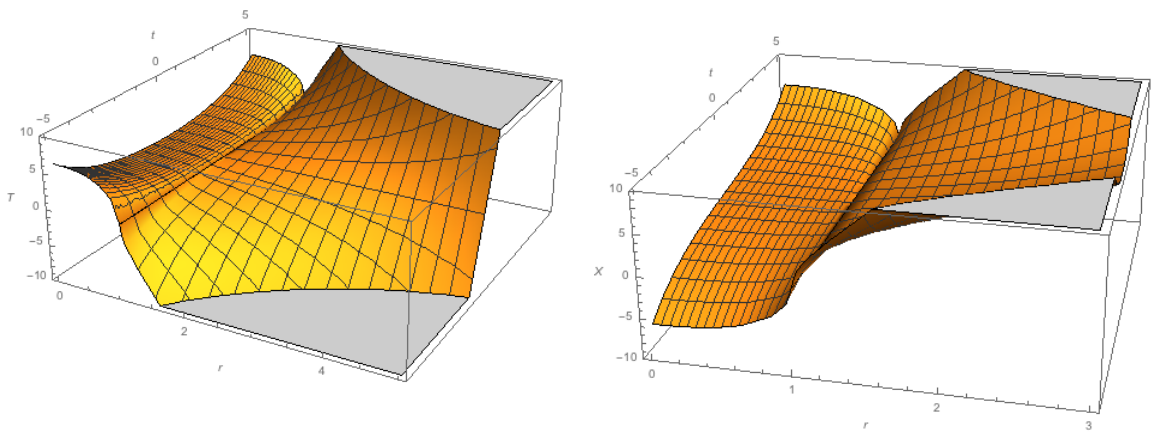
$$T := \left(1 - \frac{r}{r_s}\right)^{1/2} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right) \tag{8}$$

$$X := \left(1 - \frac{r}{r_s}\right)^{1/2} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right) \tag{9}$$

so that

$$(T, X) \in (0, \infty) \times \mathbb{R}. \tag{10}$$

Putting the two coordinates together gives us a better picture of how (T, X) depend on (t, r).



(a) $T = T(t, r)$

(b) $X = X(t, r)$

Figure 1: These two graphs plot T and X as functions of (t, r) respectively. In this figure, $r_s = 1$. The gray shaded region indicates that the graph extends beyond the specified ranges.

In this case, $(T, X) \in \mathbb{R} \times \mathbb{R}$ take values in an infinite range in both directions (subject to a constraint, which we'll get to). Using these new coordinates, we analyze particular contours that dictate certain regions of physical interest. First, notice that

$$T^2 - X^2 = \left(1 - \frac{r}{r_s}\right) e^{\frac{r}{r_s}} \tag{11}$$

for *both* domains. This gives an implicit expression of r in terms of T and X without any depenence on t. The horizon occurs at $r = r_s$ which corresponds to

$$T^2 = X^2 \tag{12}$$

i.e.

$$T = \pm X. \tag{13}$$

The singularity occurs at $r = 0$ which corresponds to

$$T^2 - X^2 = 1 \tag{14}$$

which describes a hyperbola in the (T, X) coordinate system. More generally, suppose that r is a constant. Then expression (11) is of the form $T^2 - X^2 = \pm C$.

By further manipulating the expressions for T and X, we can find other contours of interest. For instance, assuming that $T \neq 0$, we find

$$\frac{X}{T} = \tanh\left(\frac{t}{2r_s}\right). \tag{15}$$

When t is a constant, this just describes lines of constant slope. Similar arguments can be made by assuming $X \neq 0$. Therefore, lines of constant t correspond to straight lines through the origin in the (T, X) plane.

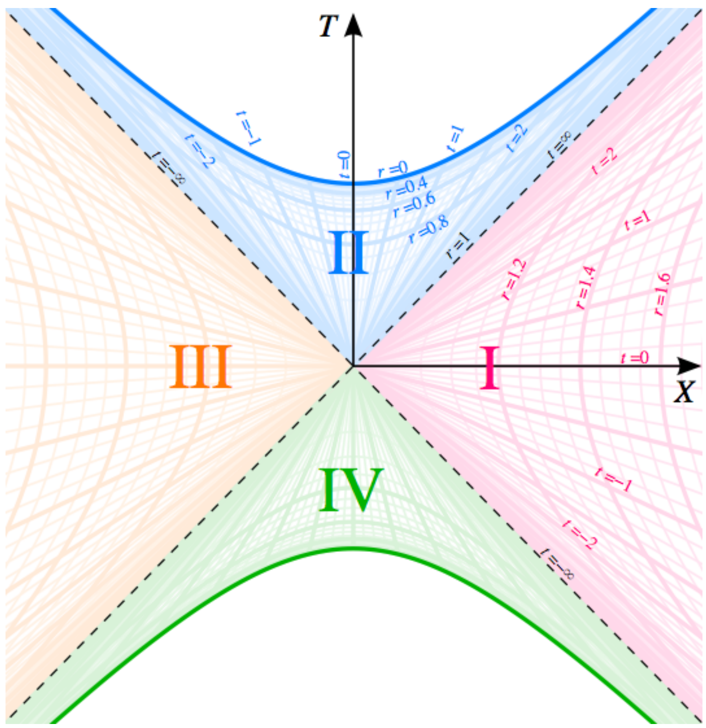


Figure 2: Some contours for the Kruskel-Szekeres coordinate chart (T, X). Picture was obtained from Wikipedia `http://en.wikipedia.org/wiki/Kruskal-Szekeres_coordinates`. The white regions inside the lower and upper hyperbolas are physically the single line corresponding to the singularity for all time.

The metric in Kruskel-Szekeres coordinates is

$$ds^2 = \frac{2r_s^3}{r} e^{-\frac{r}{r_s}} (dT^2 - dX^2). \tag{16}$$

The notation is a bit poor because the coordinate r shows up. The only reason for this is because r does not have a nice expression in terms of T and X other than the implicit equation (11). Notice that the light-cones in this coordinate system are the same at every point (except of course at $r = 0$). This is because the metric is of the form $f(dT^2 - dX^2)$ where f is some function of T and X. The causal structure, namely the family of light cones at all points of space, looks identical at every point in this coordinate system because scaling does not change these cones.

Penrose diagrams

The goal of Penrose diagrams is to find coordinates that are bounded and preserve the light-cone structure from the former set of coordinates. One way to do this is to find new coordinates that are functions of the light coordinates of the original coordinates. I don't know the precise mathematical definition, but let's go through two common examples.

Example 1. The Minkowski metric is given by

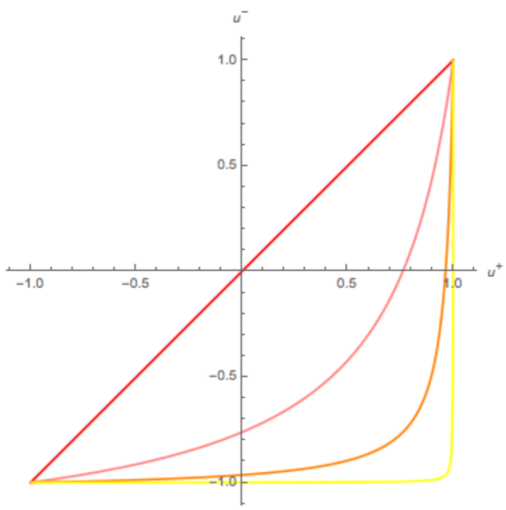
$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2. \tag{17}$$

As before, we'll ignore the spherical part. Define new coordinates

$$u^{\pm} := \tanh(t \pm r) \tag{18}$$

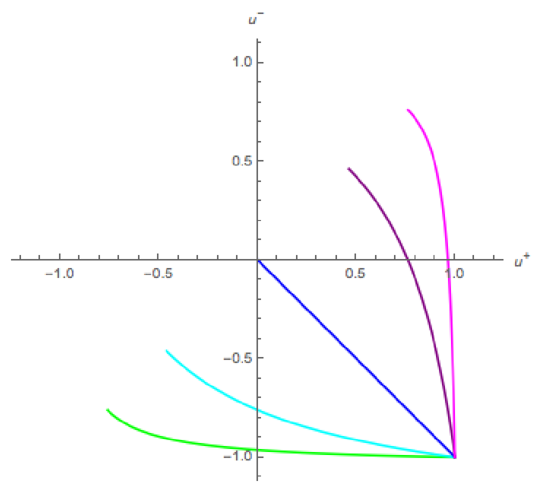
defined for all $(t, r) \in \mathbb{R} \times (0, \infty)$. Notice that the coordinates $u^{\pm}$ are confined to the square $(-1, 1) \times (-1, 1)$. It turns out a smaller region is sufficient as we'll show now. The $r = 0$ line corresponds to the line $u^+ = u^-$. The $t = 0$ line corresponds to the line $u^+ = -u^-$. All the other contours are found by using mathematica and plotting parametrized curves with varying values of fixed r and t while the other parameter runs freely.

```
r = ParametricPlot[
  {{Tanh[t], Tanh[t]},
   {Tanh[t + 0.5], Tanh[t - 0.5]},
   {Tanh[t + 1], Tanh[t - 1]},
   {Tanh[t + 2], Tanh[t - 2]}}, {t, -5, 5},
  AxesLabel → {u^+, u^-},
  PlotStyle → {Red, Pink, Orange, Yellow}]
```



(a) r has fixed values while t varies within $t \in (-5, 5)$. r increases as a function of color in the rainbow spectrum (red to yellow).

```
t = ParametricPlot[
  {{Tanh[-1 + r], Tanh[-1 - r]},
   {Tanh[-0.5 + r], Tanh[-0.5 - r]},
   {Tanh[0 + r], Tanh[0 - r]},
   {Tanh[0.5 + r], Tanh[0.5 - r]},
   {Tanh[1 + r], Tanh[1 - r]}}, {r, 0, 20},
  AxesLabel → {u^+, u^-},
  PlotStyle → {Green, Cyan, Blue, Purple, Magenta},
  PlotRange → {-1.12, 1.12}]
```



(b) t has fixed values while r varies within $r \in (0, 20)$. t increases as a function of color in the rainbow spectrum (green to blue to magenta).

Figure 3: These two graphs show the curves of constant r and constant t respectively. $t \to -\infty$ corresponds to the lower left corner (*past timelike infinity*). $t \to +\infty$ corresponds to the upper right corner (*future timelike infinity*). Therefore, if r is fixed, time moves from lower loweft to upper right. $r \to 0$ corresponds to the red line while $r \to \infty$ corresponds to the yellow curve. Therefore, distance from the black hole increases to the bottom, bottom right, and right.

Putting these two plots together gives a picture of what the constant r and constant t curves look like in the (u^+, u^-) coordinates.

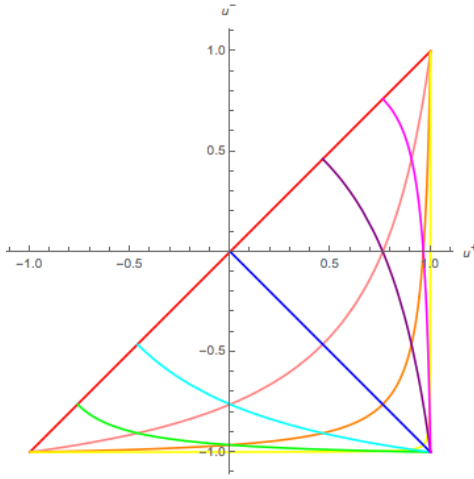
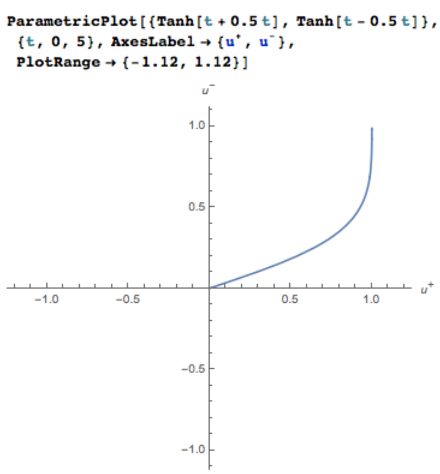
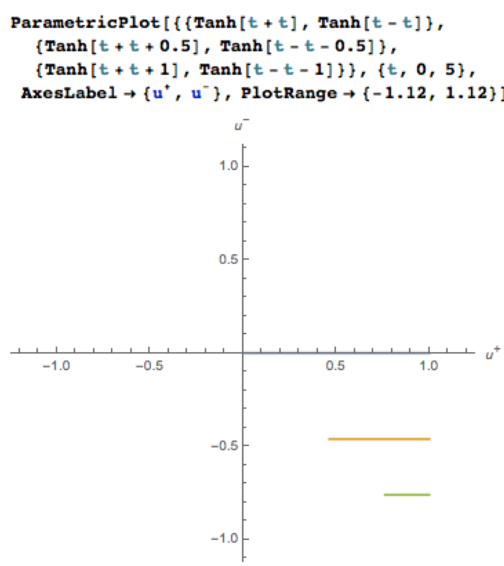


Figure 4: The two graphs from above are plotted on top of each other to show all the contours for the Penrose diagram for Minkowski space. The region inside this triangle is the Penrose diagram.

So in Minkowski space, what does a particle moving look like in a Penrose diagram? Minkowski space has no gravitational force acting on a particle so it moves along a straight line. Therefore, $r(t) = vt + r_0$ where $v \in (-1, 1)$ (remember the speed of light is set equal to 1) and $r_0 \in (0, \infty)$. To ignore the coordinate singularity at $r = 0$ (when the particle reaches the origin of our coordinate system), we can assume $v \in (0, 1)$. We can also look at what happens to a photon.



(a) The trajectory of a massive particle that begins at $r = 0$ and moves radially outward at a speed of 0.5. The particle starts off at the origin in the (u^+, u^-) plane and moves off along the path shown to the top right corner.



(b) A photon will travel directly to the right along a horizontal line. Several initial conditions are chosen for various values of initial radius.

Figure 5: Massive particle and photon trajectories shown for Minkowski metric.

This motivates the terminology that the right edge of the Penrose diagram is called the *future null infinity* because all photons approach that edge. Similarly, by reversing time, we can see where photons travel. They actually form vertical lines as the following figure shows.

```
ParametricPlot[{{Tanh[-t + t], Tanh[-t - t]},
  {Tanh[-t + t + 0.5], Tanh[-t - t - 0.5]},
  {Tanh[-t + t + 1], Tanh[-t - t - 1]}}, {t, 0, 5},
 AxesLabel → {u⁺, u⁻}, PlotRange → {-1.12, 1.12}]
```

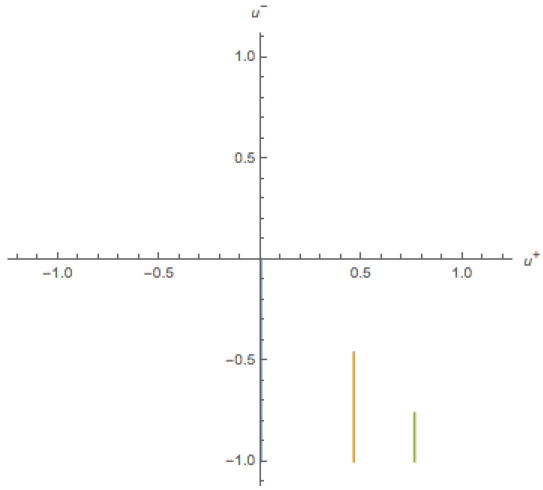


Figure 6: By reversing the t coordinate for the trajectory of a photon, we can see that the infinite past of the photon lies on the bottom edge of the Penrose diagram. We could have equivalently let time run backwards in the previous plot.

This motivates the terminology that the bottom edge of the Penrose diagram is called the *past null infinity*. There are also other terminologies that we'll mention now. If we fix $t = 0$ and vary r, then the point (technically the sphere if we go back to our usual Minkowski metric) as $r \to \infty$ is called the point (sphere) at *spatial infinity*.

Example 2. Now we go back to studying the Schwarzschild metric. This time, in the Kruske-Szekeres coordinates (T, X), the allowed values are given by

$$(T, X) \in \mathbb{R}^2 \cap C \tag{19}$$

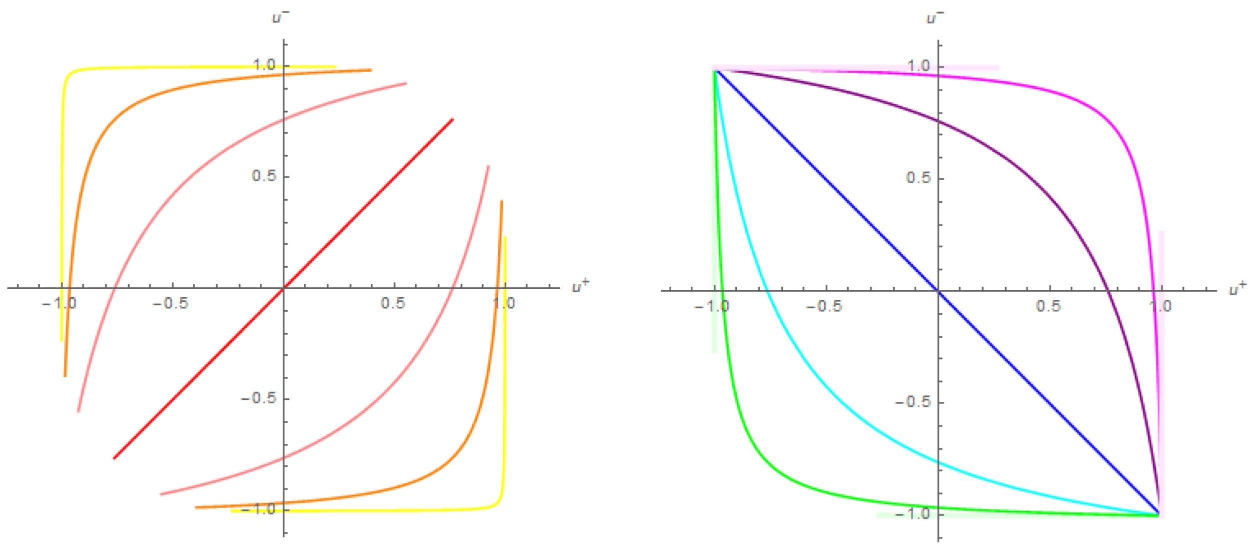
where C is the "constraint" region defined by

$$T^2 - X^2 < 1, \tag{20}$$

which is defined because of the singularity at $r = 0$ which corresponds to $T^2 - X^2 = 1$. The plots from before for the Minkowski space that were used to derive the Penrose diagram must therefore be modified. For instance, for a fixed X coordinate, the parametric plot is given by

$$(\tanh[T + X], \tanh[T - X]) \quad \text{with} \quad T \in \left[-\sqrt{1 + X^2}, \sqrt{1 + X^2}\right] \tag{21}$$

so that the domain of definition for these curves actually depends on X making it a bit more complicated. However, by separately graphing these results for various values of X the following graph is obtained. A similar graph is obtained by looking at curves of constant T and varying X.



(a) X has fixed values $X \in \{-2, -1, -0.5, 0, 0.5, 1, 2\}$ and are drawn from yellow to red back to yellow in this order while T varies within $T \in \left[-\sqrt{1+X^2}, \sqrt{1+X^2}\right]$.

(b) T has fixed values $T \in \{-2, -1, -0.5, 0, 0.5, 1, 2\}$ and are drawn from light green to blue back to light magenta in this order while X is constrained to $X^2 > T^2 - 1$.

Figure 7: These two graphs show the curves of constant X and constant T respectively.

Notice how in the above figure the curves with fixed X and varying T end at the point $\left(\tanh\left(\sqrt{1+X^2}+X\right), \tanh\left(\sqrt{1+X^2}-X\right)\right)$. This is shown more clearly in the following graph.

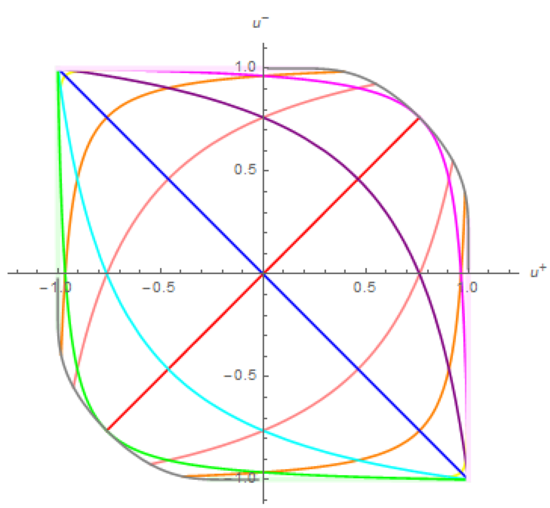
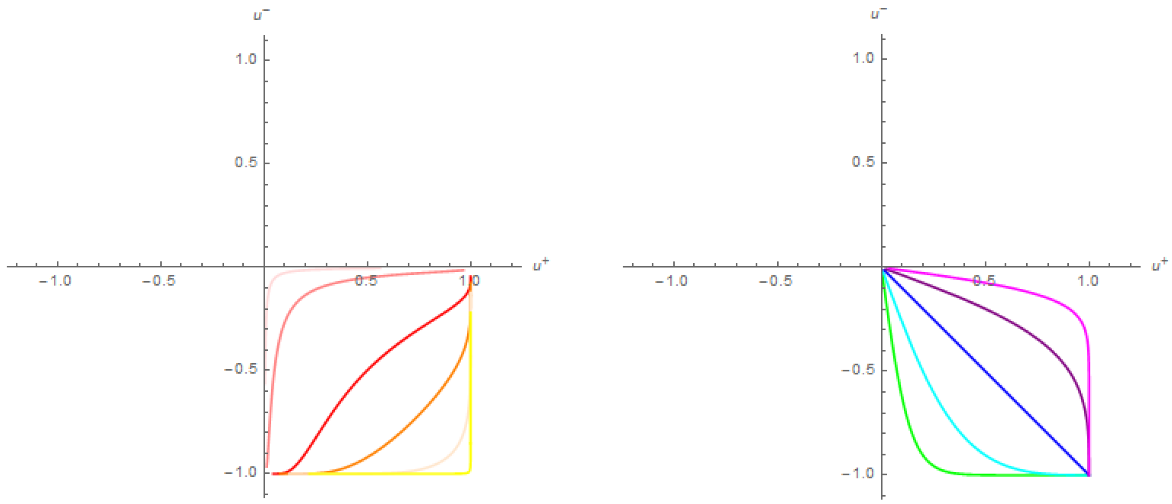


Figure 8: The two graphs from above are plotted on top of each other to show all the contours for the Penrose diagram for Schwarzshild space. The gray curves in the top right region and the bottom left region indicate the boundary of allowed values obtained by plotting $\left(\tanh\left(\sqrt{1+X^2}+X\right), \tanh\left(\sqrt{1+X^2}-X\right)\right)$ for values of $X \in (-20, 20)$. These gray curves describe $r = 0$, the black hole singularity.

It is also instructive to look at such curves in terms of the original parameters (t, r) in the Penrose diagram. This is done in the following figure (with the boundary drawn to indicate the full region missed by these coordinates). In order to do this, the expressions for $\tanh(T \pm X)$ must be written down along with their domains of definition. This is given by

$$\tanh(T \pm X) = \tanh\left[\left(\frac{r}{r_s} - 1\right)^{1/2} e^{\frac{r}{2r_s}} \left(\sinh\left(\frac{t}{2r_s}\right) \pm \cosh\left(\frac{t}{2r_s}\right)\right)\right]. \tag{22}$$



(a) $r \in \{1.001, 1.01, 1.1, 1.5, 1.5, 2\}$ while t varies within $t \in (-5, 5)$. r increases as a function of color in the rainbow spectrum (pink to yellow).

(b) t has fixed values $t \in \{-2, -1, 0, 1, 2\}$ while r varies within $r \in (0, 5)$. t increases as a function of color in the rainbow spectrum (green to blue to magenta).

Figure 9: These two graphs depict lines of constant r and t respectively for the Schwarzschild black hole. The lower right corner of this region is asymptotically Minkowski. However, as one approaches the horizon, located at the top and left edges, the curves are distorted. In these coordinates, it takes an infinite amount of time to reach the horizon.

Putting these two plots and also drawing the boundary of the Penrose diagram looks as follows.

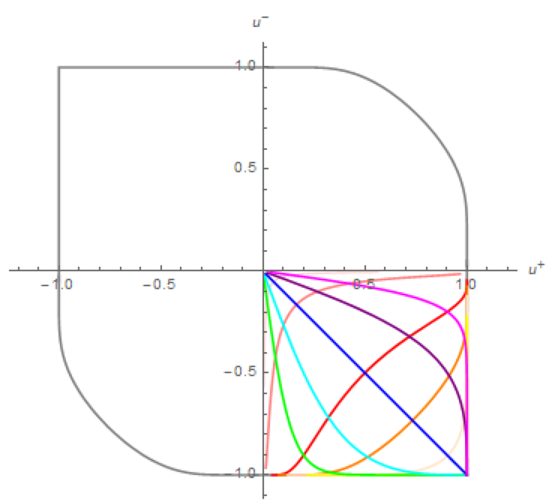


Figure 10: The Penrose diagram for the Schwarzschild black hole with respect to the (t, r) coordinates.