Categories in Probability

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Arthur J. Parzygnat

University of Connecticut, Storrs Campus
NEAM 2017 University of Albany

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   - Compact Hausdorff spaces and stochastic maps
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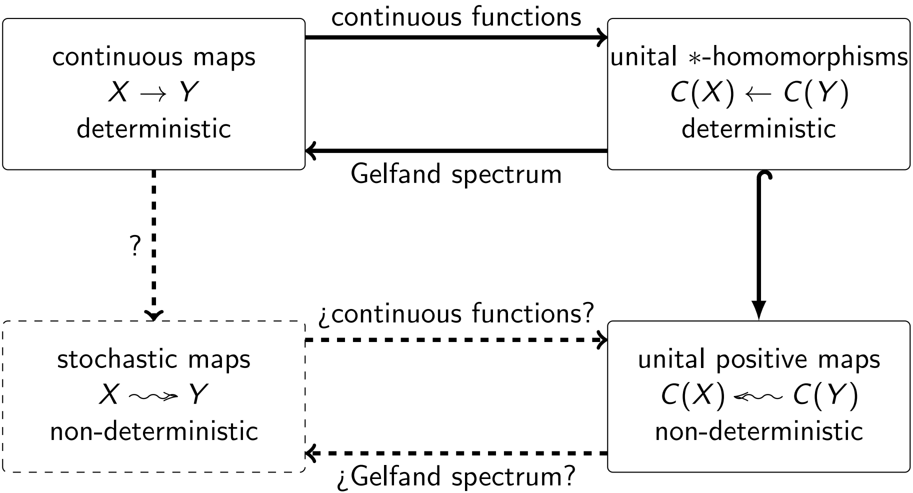
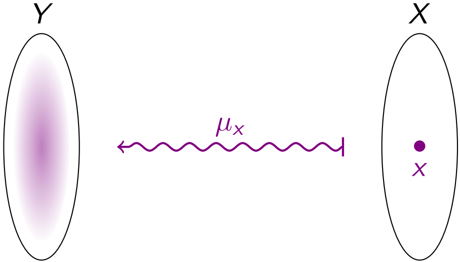
Motivation: the commutative Gelfand-Naimark Theorem



Stochastic maps

- A stochastic map $\mu : X \rightsquigarrow Y$ is a continuous function $X \to \mathrm{PM}(Y)$ from X to the space of regular probability measures on Y equipped with the vague topology.



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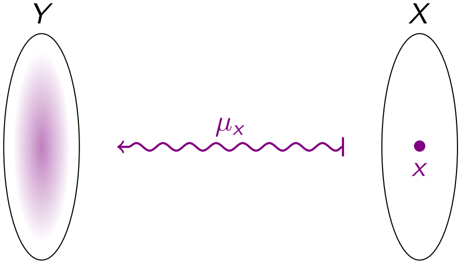

- In the vague topology on $\mathrm{PM}(Y)$, a net $(\mu_\theta)_{\theta\in\Theta}$ of measures converges to μ iff $\lim_\theta \int_Y \varphi \, d\mu_\theta = \int_Y \varphi \, d\mu$ for all $\varphi \in C(Y)$.

Composing stochastic maps

- How do you define the composition $\mu : X \rightsquigarrow Y$ followed by $\nu : Y \rightsquigarrow Z$? From $X \to \mathrm{PM}(Y)$ and $Y \to \mathrm{PM}(Z)$, you need to construct $\nu \circ \mu : X \to \mathrm{PM}(Z)$.

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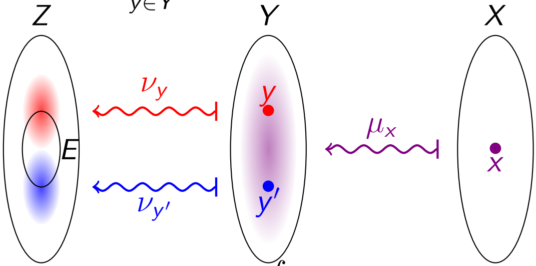
- For sets, $(\nu \circ \mu)_x(z) = \sum_{y \in Y} \nu_y(z)\mu_x(y)$. Therefore, a natural guess is



$$(\nu \circ \mu)_x(E) := \int_Y \nu_y(E)\, d\mu_x(y)$$

for all Borel sets $E \subseteq Z$. But is the *kernel* $Y \ni y \mapsto \nu_y(E)$ measurable?

Measurability of the kernel

Notice that $Y \ni y \mapsto \nu_y(E)$ is the composition

$$Y \xrightarrow{\nu} \mathrm{PM}(Z) \xrightarrow{\mathrm{ev}_E} \mathbb{R},$$

where

$$\begin{aligned} \mathrm{PM}(Z) &\xrightarrow{\mathrm{ev}_E} \mathbb{R} \\ \nu &\mapsto \nu(E). \end{aligned}$$

Hence, measurability of ev_E would imply measurability of the kernel $Y \ni y \mapsto \nu_y(E)$. So, is ev_E measurable for all Borel sets $E \subseteq Z$? Earlier proofs used separability of Z so that Z is a metric space. Instead, we suppose all probability measures are regular.

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iii. $P \subseteq L$.

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$$= \left\{ \nu \in \mathrm{PM}(Z) \ : \ \sup_{\substack{\varphi \in C(Z) \\ 0 \le \varphi \le 1_E}} \int_Z \varphi \, d\nu > a \right\}$$

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- Dynkin's π-λ theorem implies L contains the Borel sets of Z.

Compact Hausdorff spaces and stochastic maps

Let $\mu : X \rightsquigarrow Y$ and $\nu : Y \rightsquigarrow Z$ be stochastic maps. For each $x \in X$, the assignment

$$\mathrm{Borel}_Z \ni E \mapsto (\nu \circ \mu)_x(E) := \int_Y \nu_y(E)\, d\mu_x(y)$$

is a regular probability measure on Z and

$$\begin{aligned} X &\to \mathrm{PM}(Z) \\ x &\mapsto (\nu \circ \mu)_x \end{aligned}$$

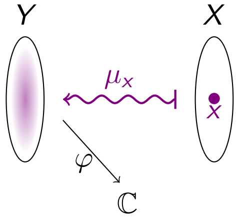
is a stochastic map. Furthermore, compact Hausdorff spaces with stochastic maps form a category.

The continuous functions functor

stochastic maps
$\mu : X \rightsquigarrow Y$
non-deterministic

continuous functions →

unital positive maps
$\int_Y \Box \, d\mu_{\Box} : C(Y) \rightsquigarrow C(X)$
$C(Y) \ni \varphi \mapsto \left(X \ni x \mapsto \int_Y \varphi \, d\mu_x \right)$
non-deterministic



The Gelfand spectrum functor

continuous maps $f^* : \sigma(\mathcal{A}) \to \sigma(\mathcal{B})$ $\mathcal{A} \ni \chi \mapsto \chi \circ f$ deterministic	$\longleftarrow$ Gelfand spectrum	unital $*$-homomorphisms $f : \mathcal{B} \to \mathcal{A}$ deterministic

$\sigma(\mathcal{A}) := \{$unital $*$-homomorphisms $\chi : \mathcal{A} \to \mathbb{C}\}$ equipped with weak* topology. Such $*$-homomorphisms are called *characters*. $\sigma(\mathcal{A})$ is called the *(Gelfand) spectrum* of $\mathcal{A}$.

Constructing the stochastic spectrum functor I

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  Let $\mathcal{S}(\mathcal{A})$ denote the space of states on $\mathcal{A}$.

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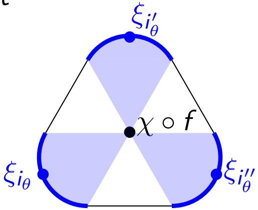
- $\mathcal{S}(\mathcal{A})$ is convex and the extreme points of $\mathcal{S}(\mathcal{A})$ are precisely the characters $\sigma(\mathcal{A})$.
- Krein-Milman implies there exists a net

$$\left(\chi_\theta = \sum_{i_\theta = 1}^{n_\theta} \lambda_{i_\theta} \xi_{i_\theta} \right)_{\theta \in \Theta}$$



in $\mathrm{convexhull}(\sigma(\mathcal{A}))$, with ξ_{i_θ} all characters, converging to $\chi \circ f$.

Constructing the stochastic spectrum functor II

- Then

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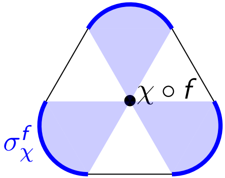

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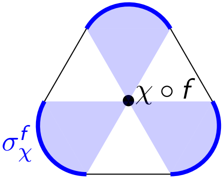

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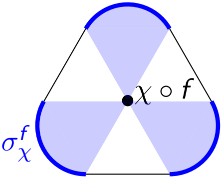

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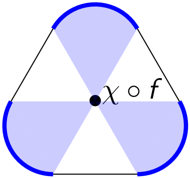

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- σ is a functor and σ together with C furnish an equivalence of categories.

Thank you

Thank you for your attention