Two-dimensional iterated integrals and applications in classical gauge theory
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All work presented here is based on arXiv:1410.6938 and a forthcoming article

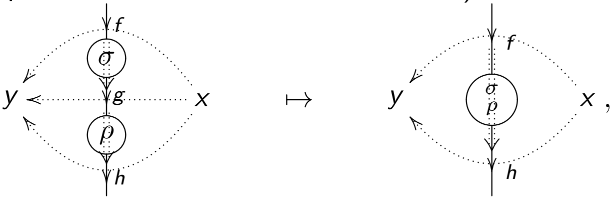
1. 2-groups
2. Principal 2-group 2-bundles
3. Local surface transport
4. Global surface transport
5. Examples

2-Groups are 2-categories I

- 2-Categories $\mathcal{C}$ have objects C_0, 1-morphisms C_1, and 2-morphisms C_2 along with functions

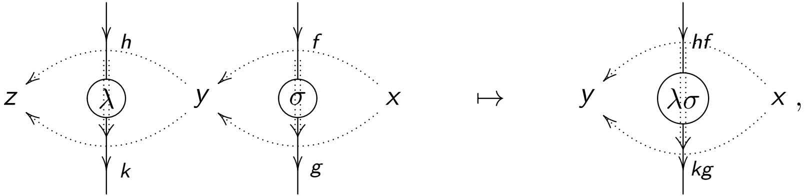
$$C_2 \underset{t}{\overset{s}{\underset{\xleftarrow{\;i\;}}{\rightrightarrows}}} C_1 \underset{t}{\overset{s}{\underset{\xleftarrow{\;i\;}}{\rightrightarrows}}} C_0$$

specifying the source and target of morphisms along with identity morphisms for every object and 1-morphism. There are also three composition maps $C_k \;{}_{s^{k-j}}\!\times_{t^{k-j}} C_k \to C_k$ for all j, k with $0 \le j < k \le 2$. The new compositions are drawn in terms of string diagrams as (dashed lines indicate the more usual 2-categorical diagram representation, i.e. the "Poincaré dual")



2-Groups are 2-categories II

and



and are both associative and unital with respect to the identity morphisms (the third composition is that of 1-morphisms, which is included in the latter case). Furthermore, they satisfy an important relation known as the interchange law.

Definition

A *2-group* is a 2-category with one object and all 1- and 2-morphisms invertible with respect to all compositions.

2-Groups are crossed modules I

As an example of a 2-group, consider the following mathematical structure, first considered by Whitehead.[1]

Definition

A *crossed module* is a quadruple (H, G, τ, α) of two groups, G and H, group homomorphisms $\tau : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$, satisfying the two conditions

$$\alpha_{\tau(h)}(h') = hh'h^{-1}$$

and

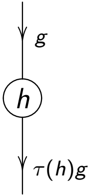
$$\tau(\alpha_g(h)) = g\tau(h)g^{-1}.$$

If the groups G and H are Lie groups and the maps τ and α are smooth, then (H, G, τ, α) is called a *Lie crossed module*.

[1] J. H. C. Whitehead. "On adding relations to homotopy groups," Ann. of Math. (2) 42 (1941) 409–428.

2-Groups are crossed modules II

A crossed module gives rise to a 2-group in the following way (in fact, all 2-groups arise in this way for some crossed module uniquely). 1-morphisms are labelled by elements g of G. A 2-morphism whose source is g and target is g' is labelled by an element h of H satisfying $g' = \tau(h)g$. Therefore, it is common to represent such a 2-morphism as

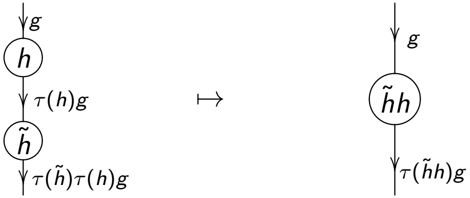


,

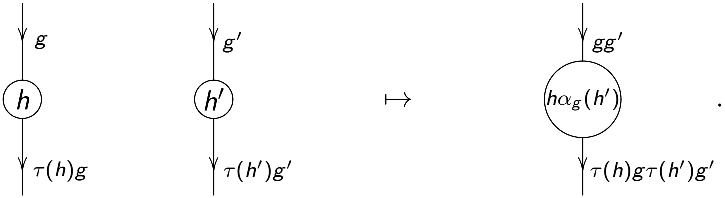
where we have dropped using the usual "globular" 2-categorical notation in terms of the more favorable string diagram notation. Notice that since a 2-group has a single object, this single object is not drawn.

2-Groups are crossed modules III

2-group compositions are written as



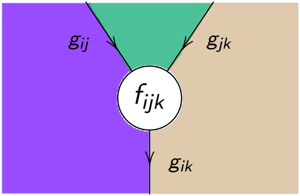
and



We will write the crossed module as $\mathcal{G}$ and the associated 2-group as $\mathcal{BG}$.

Differential cocycle data for 2-bundles I

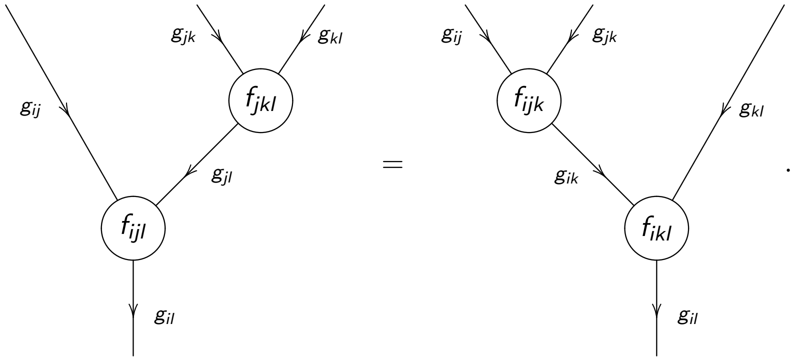
Principal $\mathcal{G}$-2-bundles over M with connection are described by "differential cocycle data" in terms of local $\mathfrak{g}$-valued 1-forms $(A_i \in \Omega^1(U_i;\mathfrak{g}))$, $\mathfrak{h}$-valued 2-forms $(B_i \in \Omega^2(U_i;\mathfrak{h}))$, transition functions $(g_{ij} \in \Omega^0(U_{ij};G))$, transition 1-forms $(\varphi_{ij} \in \Omega^1(U_{ij};\mathfrak{h}))$, and higher transition functions $(f_{ijk} \in \Omega^0(U_{ijk};H))$, for some open cover $\{U_i\}_{i\in I}$ of M. These satisfy many relations. For instance, f_{ijk} is drawn as



indicating that it interpolates between $g_{ij}g_{jk}$ and g_{ik}. Therefore, the usual cocycle condition for G-bundles $g_{ij}g_{jk} = g_{ik}$ does not hold. Instead, we have extra data, namely f_{ijk} satisfying $g_{ik} = \tau(f_{ijk})g_{ij}g_{jk}$.

Differential cocycle data for 2-bundles II

The collection of f_{ijk} must satisfy the higher cocycle condition



There are many other relations among the differential cocycle data given, too many to list here.

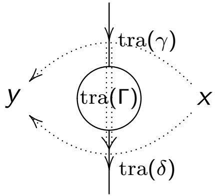
Local surface transport I

- Differential cocycle data as above give rise to, and are determined by, (weak) 2-functors $\mathrm{tra} : \mathcal{P}_2(M) \to \mathcal{B}\mathcal{G}$ with *local* (not global) smoothness properties. Here $\mathcal{P}_2(M)$ is a certain "smooth" 2-groupoid whose points are paths, 1-morphisms are thin paths, and 2-morphisms are certain equivalence classes of bigons (homotopies), called thin bigons. $\mathcal{P}_2(M)$ is called the thin path 2-groupoid of M.
- Smoothness conditions are similar to the ordinary functor case, namely there are *smooth* (strict) 2-functors $\mathrm{tra}_i : \mathcal{P}_2(U_i) \to \mathcal{B}\mathcal{G}$ related to the restriction $\mathrm{tra}|_{U_i}$ by a smooth natural isomorphism.
- The formulas in terms of iterated integrals are more complicated. There is also an equivalence of 2-categories between such differential cocycle data and smooth 2-functors as above.[2]

[2] Urs Schreiber and Konrad Waldorf. "Smooth Functors vs. Differential Forms." Homology, Homotopy Appl., 13(1):143-203, 2011. arXiv:0802.0663

Local surface transport II

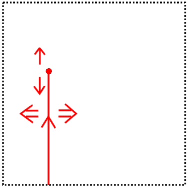
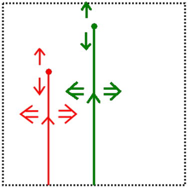
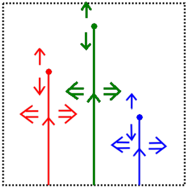
Such a smooth 2-functor $\mathrm{tra} : \mathcal{P}_2(U) \to \mathcal{BG}$ assigns 2-group elements to bigons



It is convenient to draw paths and bigons in terms of the usual globular categorical representation while decorating the paths and bigons with string diagrams when discussing transport in terms of 2-group elements. For our purposes, we will not need to know what the formula for this functor is other than the fact that on paths, it restricts to the usual one, namely, the time-ordered exponential.

Local surface transport III

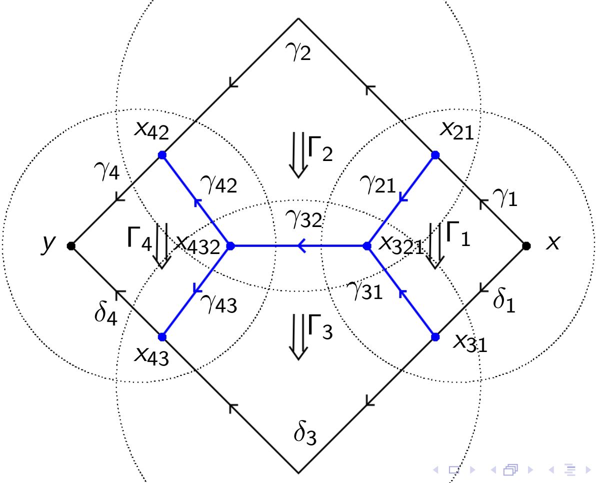
But, in case you were curious, the picture for the formula looks like the following diagram

$\mathbb{I} + \int$  $+ \int\int$  $+ \int\int\int$  $+ \cdots$

The path-ordered integral is depicted schematically as an infinite sum of terms expressed by placing B at the endpoints of the drawn paths, conjugating it by parallel transport using A. Then we use an ordinary integral over the vertical direction to get a 1-form. Finally we use the usual time-ordered integral in the horizontal direction.

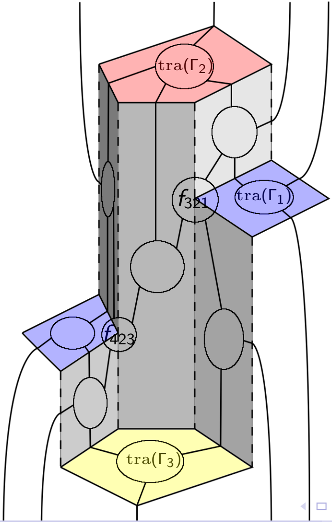
Global surface transport I

For bigons Γ not contained in a single open set, we have to choose a collection of open sets from $\{U_i\}$ that cover the bigon Γ, paths on the intersections, and points on the triple intersections.



Global surface transport II

Even in this relatively simple case, the transport is very complicated and may look something like the diagram

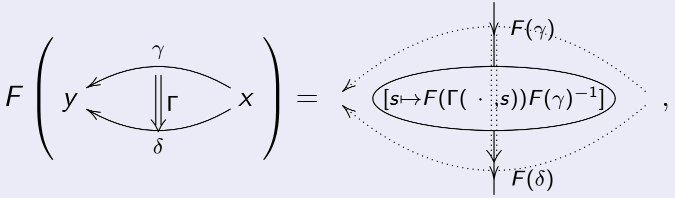


Covering 2-groups

One example of a 2-group is the universal covering 2-group where $\tau : H \to G$ is a universal cover with $H = P_e G/\sim$ (homotopy classes of paths starting at $e \in G$) and α is conjugation. Multiplication in H is by multiplying representatives. From now on, $\mathcal{G}$ will signify such a 2-group.

Theorem (P)

The formula for local parallel transport of a bigon under any smooth 2-functor $F : \mathcal{P}_2(X) \to \mathcal{B}\mathcal{G}$ is given by the formula



where $[s \mapsto g(s)]$ is the homotopy class of a path in G starting at $e \in G$ and is therefore an element of H. In particular, F is determined by the value on paths and the homotopy class of bigons.

$SU(2)$ monopoles I

A simple example of a universal covering 2-group is $\tau : SU(2) \to SO(3)$. Let $P \to S^2$ be a nontrivial principal $SO(3)$ bundle over S^2 with connection given by

$$A_N := \frac{J_3}{2}(1 - \cos\theta)d\phi \qquad \& \qquad A_S := -\frac{J_3}{2}(1 + \cos\theta)d\phi$$

and transition function

$$g_{NS}(\phi) := e^{-\phi J_3}$$

after choosing the cover to be the northern and southern hemispheres. Here

$$J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \& \quad J_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the generators of $\mathfrak{so}(3)$.

$SU(2)$ monopoles II

We define the differential cocycle data of a principal $\mathcal{G}$-2-bundle by setting

$$B_N := \frac{\sigma_3}{4i} \sin\theta \, d\theta \wedge d\phi \quad \& \quad B_S := \frac{\sigma_3}{4i} \sin\theta \, d\theta \wedge d\phi,$$

where

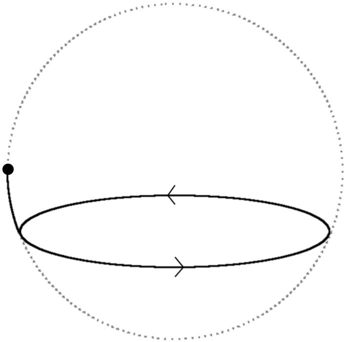
$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \& \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

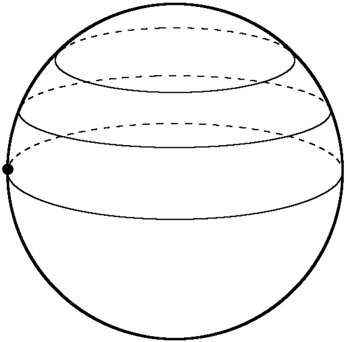
are the Pauli matrices, so that $\left\{ \frac{1}{2i}\sigma_j \right\}_{j=1,2,3}$ form a set of generators for $\mathfrak{su}(2)$. We demand that the transition 1-forms φ_{ij} vanish. All other data have been specified above. For reference, the map $\underline{\tau} : \mathfrak{su}(2) \to \mathfrak{so}(3)$ induced from the covering map on Lie algebras is given by

$$\underline{\tau}\left(\frac{1}{2i}\sigma_j \right) = J_j.$$

$SU(2)$ monopoles III

We conveniently choose the following one-parameter family of paths to describe the sphere as a bigon.

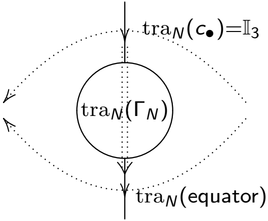


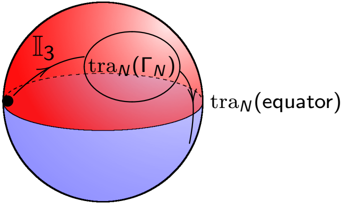


With these paths, there is no contribution from the path-ordered integral when traversing the θ direction since there is no $d\theta$ in A_N or A_S.

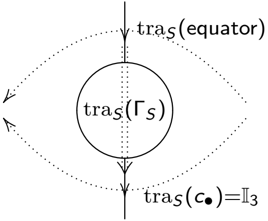
$SU(2)$ monopoles IV

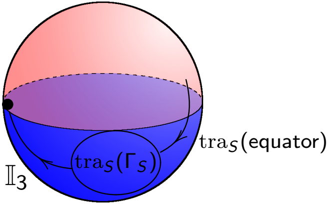
For the northern hemisphere, we have





while for the southern hemisphere, we have





$SU(2)$ monopoles V

To paste these two bigons together vertically, we have to apply the gauge transformations. First, since x is at $\phi = 0$, $g_{NS}(x) = 1$. Furthermore, since $\varphi_{NS} = 0$, the gauge transformation taking us from the northern hemisphere to the southern hemisphere is trivial (it is the identity). This is reasonable because

$$\mathrm{tra}_N(\text{equator}) = \mathrm{tra}_S(\text{equator})$$

since the lefthand side is ($\theta = \frac{\pi}{2}$ at the equator)

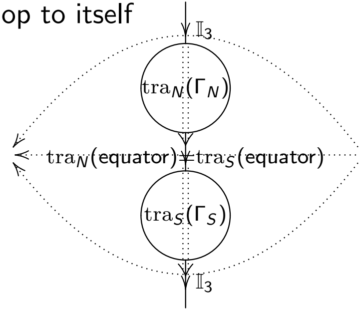
$$\mathrm{tra}_N(\text{equator}) = \mathcal{P}\exp\left\{\int_0^{2\pi} \frac{J_3}{2}\left(1 - \cos\left(\frac{\pi}{2}\right)\right) d\phi\right\} = e^{J_3\pi} = -\mathbb{I}_3$$

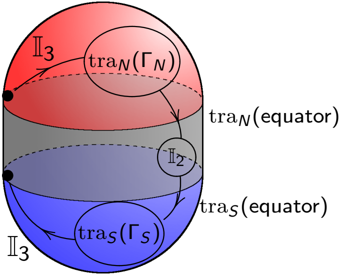
while the righthand side is

$$\mathrm{tra}_S(\text{equator}) = \mathcal{P}\exp\left\{-\int_0^{2\pi} \frac{J_3}{2}\left(1 + \cos\left(\frac{\pi}{2}\right)\right) d\phi\right\} = e^{-J_3\pi} = -\mathbb{I}_3.$$

$SU(2)$ monopoles VI

Therefore we can vertically compose both of the 2-morphisms and this gives the bigon describing the entire sphere as a bigon from the constant loop to itself





which indeed is a homotopy class of a loop based at the identity in $SO(3)$. The calculation of this loop has been known for a long time, but the above analysis gives a rigorous justification for its composition in terms of surface transport of principal 2-bundles. Nevertheless, we compute it.

$SU(2)$ monopoles VII

In the ϕ direction, the potential A only depends on J_3 and can be computed as an ordinary integral. For the bigon describing the northern hemisphere, the transport is given by the homotopy class of paths

$$\begin{aligned}\mathrm{tra}_N(\Gamma_N) &= \left[\theta \mapsto e^{\int_{\Gamma_N(\,\cdot\,,\theta)} A_N}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\theta \mapsto e^{\frac{J_3}{2}\int_0^{2\pi}(1-\cos\theta)d\phi}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\theta \mapsto e^{\pi J_3(1-\cos\theta)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}.\end{aligned}$$

Similarly, the southern hemisphere gives (the path begins at $-\mathbb{I}_3$)

$$\begin{aligned}\mathrm{tra}_S(\Gamma_S) &= \left[\theta \mapsto e^{\int_{\Gamma_S(\,\cdot\,,\theta)} A_S}(-\mathbb{I}_3)^{-1}\right]_{\theta=\frac{\pi}{2}}^{\theta=\pi} \\ &= \left[\theta \mapsto -e^{-\pi J_3(1+\cos\theta)}\right]_{\theta=\frac{\pi}{2}}^{\theta=\pi}.\end{aligned}$$

$SU(2)$ monopoles VIII

The compose these vertically, which means multiplying the representatives pointwise and taking the homotopy class, we need to make the paths have the same domain. Thus by shifting θ by $\frac{\pi}{2}$ in $\mathrm{tra}(\Gamma_S)$, we can write it as

$$\begin{aligned} \mathrm{tra}_S(\Gamma_S) &= \left[\theta \mapsto -e^{-\pi J_3\left(1+\cos\left(\theta+\frac{\pi}{2}\right)\right)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\theta \mapsto -e^{-\pi J_3(1-\sin\theta)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} . \end{aligned}$$

Finally, composing (multiplying paths) gives

$$\mathrm{tra}_S(\Gamma_S)\mathrm{tra}_N(\Gamma_N) = \left[\theta \mapsto -e^{\pi J_3(\sin\theta-\cos\theta)}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} .$$

As θ runs from 0 to $\frac{\pi}{2}$, $\sin\theta - \cos\theta$ runs from -1 to 1 monotonically. Thus, we get a loop at the identity completing exactly one full rotation along the z axis. Therefore, this loop is given by the element $-\mathbb{I}_2$, the nontrivial element in $SU(2)$ over the identity element in $SO(3)$.