# Categorical probability in the quantum realm

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IHÉS, France

Categorical Probability and Statistics 2020 June 5–8

2020 June 7

1. Introducing **fdC*-AlgU**$^{\between}$
2. Quantum and classical Markov categories
3. Subcategories of **fdC*-AlgU**$^{\between}$
4. Schwarz-positive subcategories
5. Disintegrations and Bayesian inversion

About 90% of this talk is on https://arxiv.org/abs/2001.08375 and the other 10% of this talk is based on joint work with Benjamin Russo at SUNY Farmingdale in New York and is available at https://arxiv.org/abs/1907.09689 and https://arxiv.org/abs/2005.03886.

# fdC*-AlgU$^{\between}$ as a category

The objects of **fdC*-AlgU**$^{\between}$ are finite-dimensional unital $C^*$-algebras, which are all of the form (up to isomorphism)

$$\mathcal{A} = \bigoplus_{x \in X} \mathcal{M}_{m_x},$$

where $X$ is a finite set and $\mathcal{M}_{m_x}$ is the unital $*$-algebra of all $m_x \times m_x$ matrices equipped with the operator norm and conjugate transpose as the involution $*$.

A morphism from $\mathcal{B}$ to $\mathcal{A}$ is either a linear unital map or a *conjugate* linear unital map $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$. The latter means $F(\lambda b) = \overline{\lambda} F(b)$ for all $\lambda \in \mathbb{C}$ and $b \in \mathcal{B}$, where $\overline{\lambda}$ is the conjugate transpose of $\lambda$.

# Tensor products in **fdC*-AlgU**$^{\between}$?

You can take the tensor product of linear maps to get a linear map, you can take the tensor product of conjugate linear maps to get a conjugate linear map, but you can't take the tensor product of a linear map and a conjugate linear to get one or the other. Indeed if $F$ is conjugate linear and $G$ is linear then

$$(F \otimes G)(\lambda x \otimes y) = (F \otimes G)(x \otimes \lambda y)$$

$$(F \otimes G)(\lambda x \otimes y) = F(\lambda x) \otimes G(y) = \overline{\lambda} F(x) \otimes G(y)$$

$$(F \otimes G)(x \otimes \lambda y) = F(x) \otimes \lambda G(y) = F(x) \otimes \lambda G(y)$$

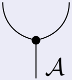
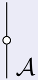
# **fdC*-AlgU**$^{\between}$ as a $\mathbb{Z}_2$-graded $\otimes$-category

Thus, **fdC*-AlgU**$^{\between}$ is not a monoidal category with the usual tensor product. Instead, we can only take the tensor product of "even" morphisms (linear maps) and "odd" morphisms (conjugate linear maps). There is also a unit $I$ equipped with an even and odd morphism that act as an identity for the $\mathbb{Z}_2$-monoidal structure.
This can be made precise with the notion of $G$-graded monoidal categories of Fröhlich and Wall.

# Quantum Markov categories

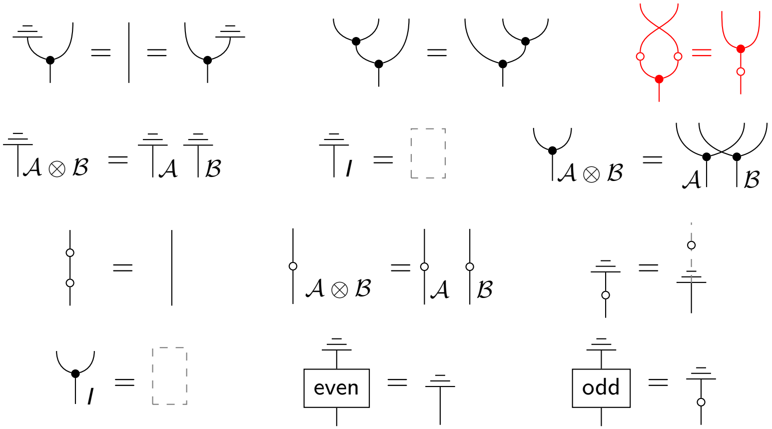
## Definition

A **quantum Markov category** (QMC) is a $\mathbb{Z}_2$-monoidal category $\mathcal{M}$ together with a family of morphisms **copy** $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \rightsquigarrow \mathcal{A}$, **discard** $!_{\mathcal{A}} : I \rightsquigarrow \mathcal{A}$, and **involve** $*_{\mathcal{A}} : \mathcal{A} \rightsquigarrow \mathcal{A}$, all depicted in string diagram notation as

$\mu_{\mathcal{A}} \equiv$  , $!_{\mathcal{A}} \equiv$  , and $*_{\mathcal{A}} \equiv$  ,

for all objects $\mathcal{A}$ in $\mathcal{M}$. These morphisms are required to satisfy several conditions.

# QMC String diagrams



# QMC String diagrams for **fdC*-AlgU**$^{\between}$

$1_{\mathcal{A}}a = a = a1_{\mathcal{A}}$ $(a_1a_2)a_3 = a_1(a_2a_3)$ $a_2^*a_1^* = (a_1a_2)^*$

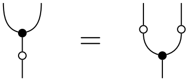
$1_{\mathcal{A}\otimes\mathcal{B}} = 1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$ $!_{\mathbb{C}}(\lambda) = \lambda$ $(a\otimes b)(a'\otimes b') = (aa')\otimes(bb')$

$(a^*)^* = a$ $(a\otimes b)^* = a^*\otimes b^*$ $(\lambda 1_{\mathcal{A}})^* = \overline{\lambda}1_{\mathcal{A}}$

$\mu_{\mathbb{C}}(\lambda\otimes\lambda') = \lambda\lambda'$ $f_{\text{even}}(\lambda 1_{\mathcal{B}}) = \lambda 1_{\mathcal{A}}$ $f_{\text{odd}}(\lambda 1_{\mathcal{B}}) = \overline{\lambda}1_{\mathcal{A}}$

# (Classical) Markov categories

If there is a subcategory $\mathcal{C}$ of $\mathcal{M}$ that is also a quantum Markov category but satisfies, in addition,



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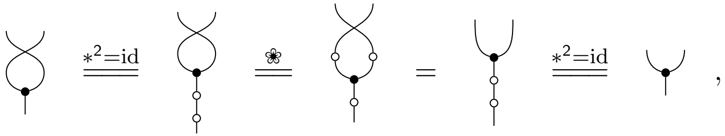
for all objects in $\mathcal{C}$, then $\mathcal{C}_{\text{even}}$ is said to be a **classical Markov subcategory** of $\mathcal{M}$. Thus,




which reproduces the usual definition of a Markov category. In the case of **fdC*-AlgU**$^{\between}$, the subcategory of *commutative* finite-dimensional $C^*$-algebras and positive unital maps (defined shortly) is equivalent to **FinStoch**$^{\text{op}}$, the category of finite sets and stochastic maps.

# Positive maps

**fdC*-AlgU**$^{\between}$ has several important subcategories.

## Definition

An element of a $C^*$-algebra $\mathcal{A}$ is **positive** iff it equals $a^*a$ for some $a \in \mathcal{A}$. A linear map $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ is **positive** iff it sends positive elements to positive elements.

## Example

For matrix algebras, a matrix is positive if and only if it is self-adjoint and its eigenvalues are non-negative. The transpose map $\mathcal{M}_m \ni A \mapsto A^T \in \mathcal{M}_m$ is positive unital.

Let **fdC*-AlgPU** denote the subcategory of **fdC*-AlgU**$^{\between}$ consisting of the same objects as **fdC*-AlgU**$^{\between}$ but the morphisms are only all the positive unital (PU) maps.

# Schwarz positive maps

**Definition**

A linear map $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ is **Schwarz positive** (SP) iff it satisfies $F(b^*b) \geq \|F(1_{\mathcal{B}})\| F(b)^*F(b)$ for all $b \in \mathcal{B}$.

Every Schwarz positive map is positive, but the converse is not true!

**Example**

The map $F : \mathcal{M}_2 \rightsquigarrow \mathcal{M}_2$ given by taking the transpose, namely $F(b) := b^T$, is positive unital but not Schwarz positive. Indeed, $F(b^*b) = \left(\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\right)^T = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$, while $F(b)^*F(b) = \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$.

Let **fdC*-AlgSPU** denote the subcategory of **fdC*-AlgPU** consisting of the same objects as **fdC*-AlgPU** but the morphisms are only all the Schwarz positive unital (SPU) maps.

# Completely positive maps

## Definition

A linear map $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ is *n*-**positive** iff $\mathrm{id}_{\mathcal{M}_n} \otimes F : \mathcal{M}_n \otimes \mathcal{B} \rightsquigarrow \mathcal{M}_n \otimes \mathcal{A}$ is positive. $F$ is **completely positive** iff $F$ is $n$-positive for all $n \in \mathbb{N}$.

Every $n$-positive map is $(n-1)$ positive, and every 2-positive map is Schwarz positive, but not every Schwarz positive map is 2-positive!

## Example (Choi 1980)

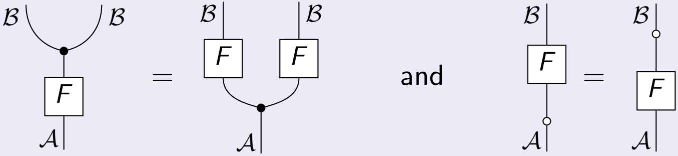
The map $F : \mathcal{M}_2 \rightsquigarrow \mathcal{M}_2$ given by $F(b) := \frac{1}{2}b^T + \frac{1}{4}\mathrm{tr}(b)\mathbb{1}_2$ is Schwarz positive unital but not 2-positive.

Let **fdC*-AlgCPU** denote the subcategory of **fdC*-AlgSPU** consisting of the same objects as **fdC*-AlgSPU** but the morphisms are only all the completely positive unital maps.

# The deterministic subcategory

## Definition

A linear map $F : \mathcal{B} \to \mathcal{A}$ is a $*$-**homomorphism** (or **deterministic**) iff $F(bb') = F(b)F(b')$ and $F(b)^* = F(b^*)$. In string diagrams:



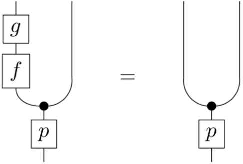

All $*$-homomorphisms are completely positive, but there are completely positive maps that are not $*$-homomorphisms. Let **fdC*-AlgDU** be the subcategory of **fdC*-AlgCPU** consisting of deterministic unital maps only. Thus, we have a hierarchy of notions of positivity.

$$\textbf{fdC*-AlgDU} \subseteq \textbf{fdC*-AlgCPU} \subseteq \textbf{fdC*-AlgSPU} \subseteq \textbf{fdC*-AlgPU}.$$

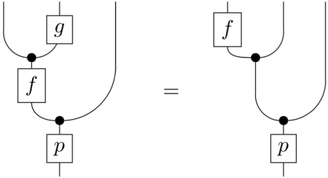
# Fritz' definition of a positive Markov category

In his first draft of "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics," Fritz defined a positive Markov category:

**11.19. Definition.** *We say that* C *is* positive *if the following condition holds: whenever*




*holds for any morphisms as indicated, then also*

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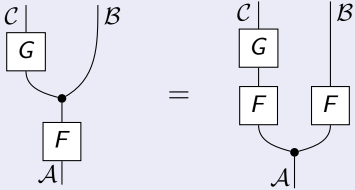
# Fritz' definition of a positive Markov category

I begun my work on the relationship between disintegrations and Bayesian inversion roughly in May 2019. When Fritz' paper came out in August, I immediately tried checking if **fdC*-AlgCPU** was positive (as a subcategory of **fdC*-AlgU**$^{\between}$). But I couldn't prove it! Two months later, Fritz had updated his definition (which I just adapted to the QMC context):

## Definition

Let $\mathcal{M}$ be a quantum Markov category. A subcategory $\mathcal{P} \subseteq \mathcal{M}_{\text{even}}$ is said to be **S-positive** in $\mathcal{M}$ iff for every pair of composable morphisms $\mathcal{C} \overset{G}{\rightsquigarrow} \mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ in $\mathcal{P}$ such that $F \circ G$ is deterministic, then



# **fdC*-AlgSPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$

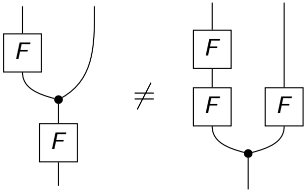
Why am I calling this condition S-positivity instead of Fritz' terminology of just positivity?

**Theorem (P. 2001.08375 [quant-ph])**

**fdC*-AlgSPU** *is an S-positive subcategory of* **fdC*-AlgU**$^{\between}$*.*

But... **fdC*-AlgPU** is *not* a positive subcategory of **fdC*-AlgU**$^{\between}$! Indeed, the transpose map $F : \mathcal{M}_m \rightsquigarrow \mathcal{M}_m$ composed with itself is the identity, and is therefore deterministic, but

$$a \otimes b \mapsto a^T b \mapsto b^T a$$




$$a \otimes b \mapsto a \otimes b^T \mapsto ab^T$$

# **fdC*-AlgSPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$

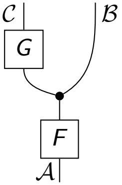
I think the proof is neat so let's try it out. It rests on something called the "Multiplication Theorem" for Schwarz positive maps.

**Lemma (The Multiplication Theorem)**

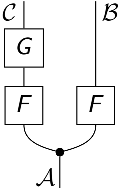
*Let* $\mathcal{B} \overset{\varphi}{\rightsquigarrow} \mathcal{A}$ *be an SPU map between* $C^*$*-algebras. Suppose that* $\varphi(b^*b) = \varphi(b)^*\varphi(b)$ *for some* $b \in \mathcal{B}$*. Then*

$$\varphi(b^*c) = \varphi(b)^*\varphi(c) \quad \text{and} \quad \varphi(c^*b) = \varphi(c)^*\varphi(b) \qquad \forall\, c \in \mathcal{B}.$$

Now, our goal is to prove




$$F\big(G(c)b\big) \quad = \quad F\big(G(c)\big)F(b)$$



# **fdC*-AlgSPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$

Let $\mathcal{C} \overset{G}{\rightsquigarrow} \mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ be a pair of composable SPU maps of $C^*$-algebras such that the composite $F \circ G$ is a $^*$-homomorphism. Then,

$$\begin{aligned} F\big(G(c)^*G(c)\big) &\le F\big(G(c^*c)\big) && \text{by S-positivity of } G \\ &= F\big(G(c)\big)^*F\big(G(c)\big) && \text{since } F \circ G \text{ is deterministic} \\ &\le F\big(G(c)^*G(c)\big) && \text{by S-positivity of } F \end{aligned}$$

holds for all $c \in \mathcal{C}$. Thus, all inequalities become equalities. In particular,

$$F\big(G(c)^*G(c)\big) = F\big(G(c)\big)^*F\big(G(c)\big) \qquad \forall\, c \in \mathcal{C}.$$

By the Multiplicative Theorem, this implies

$$F\big(G(c)^*b\big) = F\big(G(c)\big)^*F(b) \qquad \forall\, c \in \mathcal{C},\ b \in \mathcal{B}.$$

Since $F$ and $G$ are $*$-preserving (natural with respect to $*$) and $*$ is an involution, this reproduces the required condition.

# **fdC*-AlgCPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$

**fdC*-AlgCPU** is also an S-positive subcategory of **fdC*-AlgU**$^{\between}$ (in fact, the subcategory of $n$-positive unital maps is as well for all $n \geq 2$).

**Question**

*Is* **fdC*-AlgSPU** *the largest S-positive subcategory of* **fdC*-AlgU**$^{\between}$*?*

**Question**

*Are there diagrammatic axioms that characterize the subcategory* **fdC*-AlgPU** *of positive unital maps inside* **fdC*-AlgU**$^{\between}$*?*

**Question**

*Which subcategories of* **fdC*-AlgU**$^{\between}$ *obey Fritz' first (before v. IV) notion of positive subcategory?*

# **fdC*-AlgCPU** is an S-positive $\otimes$-subcat of **fdC*-AlgU**$^{\between}$

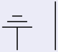
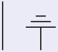
Since CP maps are S-positive, **fdC*-AlgCPU** is an S-positive subcategory of **fdC*-AlgU**$^{\between}$. Unlike **fdC*-AlgSPU**, however, **fdC*-AlgCPU** is closed under the tensor product. Thus, **fdC*-AlgCPU** is an S-positive *monoidal* subcategory of **fdC*-AlgU**$^{\between}$.

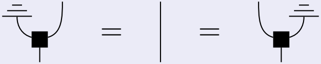
**Question**

*Is* **fdC*-AlgCPU** *the largest S-positive monoidal subcategory of* **fdC*-AlgU**$^{\between}$*?*

# A no-cloning theorem for S-positive subcategories

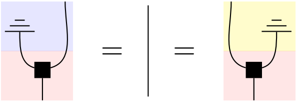
**Theorem (No broadcasting for S-positive subcategories)**

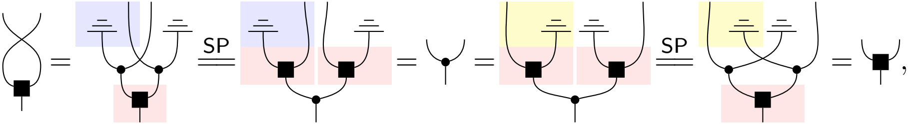
*Let $\mathcal{P}$ be an S-positive subcategory of a quantum Markov category $\mathcal{M}$ containing the morphisms* , , *and*  *for each object in $\mathcal{P}$. In addition, suppose that $\mathcal{P}$ contains a morphism*  *satisfying*

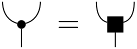



*for every object in $\mathcal{P}$. Then*  *is commutative and in fact equals duplication for every object of $\mathcal{P}$.*

# Proof of no-cloning for S-positive subcategories

Since , which is deterministic, S-positivity gives




which reproduces the commuting axiom since .

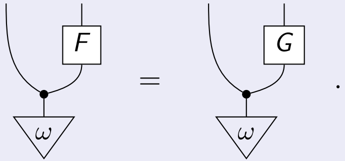
# Almost everywhere equivalence $F \underset{\omega}{=} G$

**Theorem (P. 2001.08375 [quant-ph])**

*Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras, let $F, G : \mathcal{B} \rightsquigarrow \mathcal{A}$ be two linear maps, and let $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ be a state (a PU map). Then the following are equivalent.*

i. $F(b) - G(b)$ *is in the null space* $\mathcal{N}_\omega := \{a \in \mathcal{A} : \omega(a^*a) = 0\}$ *of* $\omega$ *for all* $b \in \mathcal{B}$.

ii.

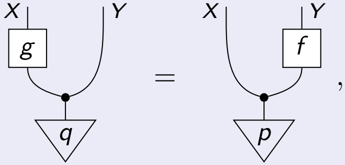



In this case, $F$ is said to be $\omega$-**a.e. equivalent to** $G$. The first definition appears in 1907.09689 [quant-ph] and the second (for ordinary Markov categories) is due to Cho–Jacobs 1709.00322 [cs.AI].

# Bayes' theorem

## Theorem (Bayes' theorem)

*Let $X$ and $Y$ be finite sets, let $\{\bullet\} \stackrel{p}{\rightsquigarrow} X$ be a probability measure, and let $X \stackrel{f}{\rightsquigarrow} Y$ be a stochastic map. Then there exists a stochastic map $Y \stackrel{g}{\rightsquigarrow} X$ such that*




*where $\{\bullet\} \stackrel{q}{\rightsquigarrow} Y$ is given by $q := f \circ p$. Furthermore, for any other $g'$ satisfying this condition, $g \underset{q}{=} g'$.*

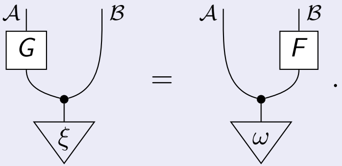
You can watch my video explaining *why* I call this Bayes' theorem here.

# Bayesian inverses

The previous theorem motivates the following definition.

## Definition

Let $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ be a CPU map, let $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ be a state, and set $\xi := \omega \circ F$. A **Bayesian inverse** of $F$ is a CPU map $\mathcal{A} \overset{G}{\rightsquigarrow} \mathcal{B}$ such that




The existence of Bayesian inverses is not guaranteed for CPU maps between finite-dimensional $C^*$-algebras. A linear algebraic theorem characterizing its existence in **fdC*-AlgCPU** is given in 2005.03886 [quant-ph] (joint with Russo).

# Properties of Bayesian inversion

Nevertheless, when they exist, Bayesian inverses satisfy many convenient properties.

## Theorem

*Let $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$, $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$, and $\xi := \omega \circ F$ be as before.*

i. *If $G$ is a Bayesian inverse of $(F, \omega)$, then $\omega = \xi \circ G$.*
ii. *If $G$ is a Bayesian inverse of $(F, \omega)$, then it is necessarily $\xi$-a.e. unique.*
iii. *If $F$ is a $^*$-isomorphism, then $G = F^{-1}$ is a Bayesian inverse of $(F, \omega)$.*
iv. *The composite of Bayesian inverses is a Bayesian inverse of the composite.*
v. *A Bayesian inverse of a Bayesian inverse is a.e. equivalent to the original map.*
vi. *A tensor product of Bayesian inverses is a Bayesian inverse of the tensor product.*

# Disintegrations

## Definition

Let $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$, $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$, and $\xi := \omega \circ F$ be as before. A **disintegration** of $(F, \omega)$ is a CPU map $\mathcal{A} \overset{G}{\rightsquigarrow} \mathcal{B}$ such that

$$\xi \circ G = \omega \qquad \text{and} \qquad G \circ F \underset{\xi}{=} \mathrm{id}_{\mathcal{B}}.$$

## Theorem (P. 2001.08375 [quant-ph])

*Let $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$, $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$, and $\xi := \omega \circ F$ be as before.*

i. *If $(F, \omega)$ has a disintegration, then $F$ is $\omega$-a.e. deterministic (see paper for definition).*
ii. *If $(F, \omega)$ has a disintegration $G$, then $G$ is a Bayesian inverse of $(F, \omega)$.*
iii. *If $F$ is deterministic and $(F, \omega)$ has a Bayesian inverse $G$, then $G$ is a disintegration of $(F, \omega)$.*

All this and much more can be found in the following references.


- K. Cho and B. Jacobs "Disintegration and Bayesian Inversion via String Diagrams" 1709.00322 [cs.AI]
- T. Fritz "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics" 1908.07021 [math.ST]
- A. Parzygnat and B. Russo "Non-commutative disintegrations: existence and uniqueness in finite dimensions" 1907.09689 [quant-ph]
- A. Parzygnat "Inverses, disintegrations, and Bayesian inversion in quantum Markov categories" 2001.08375 [quant-ph]
- A. Parzygnat and B. Russo "A non-commutative Bayes' theorem" 2005.03886 [quant-ph]
- A. Parzygnat "Categorical probability theory" videos available at https://www.youtube.com/playlist?list=PLSx1kJDjrLRSKKHj4zetTZ45pVnGCRN80


# Thank you!