SCRATCH NOTES FOR QUANTUM MECHANICS

ARTHUR J. PARZYGNAT

Contents

The following lectures are largely based on Brian C. Hall's book *Quantum Theory for Mathematicians* published by Springer [6]. In fact, much of the presentation follows that book and in many parts, statements are simply copied. The reader is urged to look up the details in that text, which is quite well-written. These lectures are also based off my learning as a physicist and the physicists who have taught me. Additional references are given in the body whenever used. My aim is not to prove many theorems. It is merely to state facts that I find interesting and useful for my understanding of physics. These are basically my personal notes that I'm just making available to others.

These lectures were given over the course of three semesters at the mathematical physics seminar including Spring 2014, Fall 2014, and Spring 2015. The author was supported by the National Science Foundation Graduate Research Fellowship under Grant No. 40017-06-06 during a part of this process.

1. First lecture: history and axioms of quantum mechanics

Since the first lecture will be largely historical, there will be almost no rigor. The following is mostly a story.

1.1. **Some comments on physics.** Our experience has lead us to discover classical physics, a field of physics by and large agreeing with our intuitions of logic. This logic stems from our experience as macroscopic beings living on Earth and interacting with other macroscopic beings. Our macroscopic sizes (with respect to atomic scales) allow us senses that give us the ability to detect distance, weight, time and so on in a way sufficient for our survival. These scales fall into a certain range, one which is experienced by macroscopic organisms like ourselves. For example, our heartbeat gives us an idea of time on a small scale while the sun gives us an idea of time on a large scale. Classical mechanics is based off of these senses. Our concept of Galilean and Newtonian relativity, gravity on earth and beyond, and electromagnetism are the common cases where our senses allowed us to extract physical laws that predict certain phenomena which can be experimentally confirmed. I'm not saying these realizations are unimportant—they are incredibly important in that they seem to dictate some order in our universe, an order which is comprehendible by us via logic and mathematics.

However, because classical physics is based off our senses and notions of logic, we should be careful to extrapolate these laws to scales beyond our typical senses. We should be careful about microscopic objects, smaller time scales, lighter masses, and smaller distances. Dually, we should also be careful about assuming our classical laws hold for larger scales. Quantum physics is one formulation of physical law that allows us to go beyond our typical scales. Again, we should only assume it holds on some scales, and in fact it is known to not accurately describe physics on other scales (such as high velocities). Our current belief, held by some scientists, is that whenever we discover a physical theory, it holds on some range of scales (provided this notion even makes sense) and any theory which claims to describe all of physics for some range of scales should reduce to the well-known theories at the scales included in this range for which those theories accurately describe our universe. Whether or not a truly universal theory holds on all scales is beyond what we understand.

1.2. **Classical mechanics.** For the duration of this lecture, we will be concerned with classical mechanics restricted to flat space, i.e. $\mathbb{R}^n$. I will not be mathematically precise because to do so would require an entire lecture or more.

Classical mechanics deals with structureless (and distinguishable) particles, rigid bodies, waves, and fields. For our purposes, we will focus on the classical physics of particles. Small particles are assumed to be represented by mathematical points in $\mathbb{R}^3$ since space is "obviously" three-dimensional and $\mathbb{R}^3$ is the simplest such space. Newton taught us that what determines the motion of a particle is a force and that this force is proportional to the particle's acceleration through the equation $F = ma$, where F is the net external force, a is the acceleration, and m is the particles mass. Another way to write this is $F = \frac{dp}{dt}$, where p is the momentum $p = mv$, where v is the velocity (mass is assumed to be independent of time), and $\frac{d}{dt}$ is the derivative with respect to time, supposing that the particle travels in time. The energy of a particle is typically given as a sum of kinetic and potential

terms $H(x,p) = \frac{p^2}{2m} + V(x)$. As a function of x and p, H is called the classical Hamiltonian. In general, the Hamiltonian is an example of a *classical observable*, which in general is some function of x and p. Newton's law can be expressed as a set of first order partial differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \tag{1}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} \tag{2}$$

called Hamilton's equations. One can check explicitly that when H is of the form mentioned above that these equations reduce to $F = ma$, where $F = -\frac{dV}{dx}$. However, Hamilton's equations hold more generally for arbitrary Hamiltonians in arbitrary coordinates. The collection of x's and p's form what is called a *classical phase space*. The set of functions on phase space, i.e. classical observables, actually forms a Lie algebra with the Poisson bracket defined by

$$\{f,g\} := \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial x}, \qquad \text{where } f,g \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3) \tag{3}$$

and where $\cdot$ represents the usual inner product on $\mathbb{R}^3$. Recall, a Lie algebra is a vector space with a binary product $\{\,\cdot\,,\,\cdot\,\}$ that satisfies the following properties

i) $\{f,g+ch\} = \{f,g\} + c\{f,h\}$ for all $f,g,h \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ and $c\in\mathbb{R}$
ii) $\{f,g\} = -\{g,f\}$ for all $f,g \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$
iii) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ for all $f,g,h \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ (this is called the Jacobi identity)
iv) $\{f,gh\} = g\{f,h\} + \{f,g\}h$ for all $f,g,h \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ (this means that $\{\,\cdot\,,\,\cdot\,\}$ is a derivation over the product of functions)

Technically, this last property holds only if the vector space is in also an associative algebra. Therefore, conditions (1)-(3) specify what a Lie algebra is.

One of the important properties of quantum mechanics will be related to the Poisson bracket. It is therefore important to stress special examples of functions. The six projection maps $\mathbb{R}^6 \to \mathbb{R}$ are (abusively) written as x_i and p_i with $i = 1,2,3$. These functions satisfy the *canonical Poisson bracket relations*

$$\{x_i, x_j\} = 0 \tag{4}$$

$$\{p_i, p_j\} = 0 \tag{5}$$

$$\{x_i, p_j\} = \delta_{ij} \tag{6}$$

for all $i,j = 1,2,3$.

And yet another important reason for introducing Poisson bracket is the following fact. If $(x(t),p(t))$ is a solution to Hamilton's equations and f is any function on phase space then along the trajectory of the solution

$$\frac{d}{dt}f(x(t),p(t)) = \{f,H\}(x(t),p(t)) \tag{7}$$

which is more commonly written as

$$\frac{df}{dt} = \{f,H\}. \tag{8}$$

This approach is more useful for classical mechanics on arbitrary manifolds since it gives a coordinate independent way to describe time evolution. Notice that in particular, one obtains Hamilton's equations from this equation.

1.3. **Problems with classical physics.** One of the major problems with classical physics and its predictions came from the "ultraviolet catastrophe" in the context of blackbody radiation. A blackbody is a perfect absorber of electromagnetic radiation (light). If electromagnetic radiation is pumped into a blackbody cavity, the energy fluctuates the atoms of the cavity and causes infrared radiation to be emitted until there is thermal equilibrium, which occurs when absorption balances the emission. This occurs for all frequencies of electromagnetic radiation. In thermal equilibrium at temperature T, the average kinetic energy per molecule is given by $\frac{kT}{2}$, where k is Boltzman's constant ($k = 1.38 \times 10^{-23}$ Joules per Kelvin). This is called the equipartition theorem. In our case, what takes place of the molecule is a standing electromagnetic wave in the cavity. The total energy in this case is twice the kinetic energy (this is a common fact of systems executing simple harmonic motion as a wave does). Thus, the energy is kT and this is *independent* of the frequency of radiation. The energy density per unit frequency is given by the Rayleigh-Jeans formula

$$\rho_T(\nu) = \frac{8\pi\nu^2 kT}{c^3}, \tag{9}$$

where ν is the frequency. For low frequency, this matches experimental results. However, for high frequencies, the formula is quadratic and diverges. This is in sharp contrast to experiment which showed that there is actually a decay for high frequencies.

In 1900, Planck realized that classical laws did not accurately account for certain properties of radiant heat. He was forced to assume that the energy is not indefinitely divisible but instead must come in discrete units instead of a continuous flow. Although Planck did not introduce this "quantization" for the problem of blackbody radiation [9], it's application is a simple consequence. By assuming that the energy of the standing electromagnetic waves in the cavity were quantized, one arrives at the Planck formula

$$\rho_T(\nu) = \frac{8\pi\nu^2 kT}{c^3} \frac{hv}{e^{\frac{h\nu}{kT}} - 1}, \tag{10}$$

where h is Planck's constant

$$h = 6.63 \times 10^{-34} \text{ Joules times seconds.} \tag{11}$$

This formula agrees with experiment. For more details, one can look at [3].

In 1905, Einstein wrote a paper on the photoelectric effect. The photoelectric effect occurs when electromagnetic radiation strikes a metal causing electrons to be emitted. Specifically, what is observed is that increasing the intensity does not increase the energy of the emitted electron (as one would guess from the classical theory), but only increases the *number* of electrons emitted. Einstein's solution to this problem was to assume that light consists of individual particles which transfer their energy to individual electrons. If light is thought of as a particle, called a *photon*, then one imagines increasing the intensity of light as corresponding to increasing the number of photons as opposed to increasing their frequency (energy).

In 1888, Rydberg tried to explain why when electricity is passed through hydrogen gas, light of very specific frequency is emitted. The energy from the electricity was transferred to the electrons and the electrons emit light of a certain discrete frequency as opposed to a continuous range of frequencies. This frequency is related

to the energy emitted by the electron. Rydberg postulated that the form of this energy is

$$E_n = -\frac{R}{n^2}, \qquad \text{where} \quad n \in \mathbb{Z}^+ \tag{12}$$

and where R is Rydberg's constant, which turns out to be given by

$$R = \frac{2\pi^2 m_e Q^4}{h^2}, \tag{13}$$

where Q is the charge of the electron and m_e is the mass of the electron. The frequencies of light emitted are of the form

$$\omega = \frac{2\pi}{h}(E_n - E_m) \tag{14}$$

for some $n > m$.

In 1913, Niels Bohr looked at the spectrum of hydrogen and tried to explain it "semi-classically" by picturing an electron orbiting the nucleus but postulating that the angular momentum of the electron was quantized. This forced the radius of the orbit to be given discretely by

$$r_n = \frac{n^2 h^2}{4\pi^2 m_e Q^2} \tag{15}$$

The energy associated to such motion at these fixed radii exactly gave Rydberg's formula for the energy spectrum of hydrogen. In 1924, Louis de Broglie realized that if the electron is described as a wave with its wavelength λ related to its momentum p by

$$p = \frac{h}{2\pi\lambda} \tag{16}$$

then such a wave would require discrete radii. This is because any given orbit consist of an integral number of periods of that wave. The major result of de Broglie's work was that what we thought of as certain objects having particle-like properties is misconstrued and in fact there are certain wave-like aspects to these objects as well.

The point of this brief history is to illustrate that quantum theory, as it is currently formulated, was certainly not obvious. It took many years of work. There were several other experiments done which I have not mentioned.

1.4. **Postulates of quantum mechanics.** Although the previous discussion will not immediately seem related to the following postulates of quantum mechanics, the relationship will hopefully be made slightly more clearer in the next section. It took over 25 years to finalize these postulates, so it should not be obvious at all that they work. The surprising thing, in fact, is that they do work to excellent accuracy. What's more surprising is that the language used to describe it is inherently mathematical.

Heisenberg put forth his ideas on matrix mechanics in 1925 (at approximately the age of 24!). These ideas stemmed largely from the quantization aspects such as spectral lines emitted from molecules such as hydrogen. We already saw that the frequencies of light emitted from hydrogen are of the form $\omega = \frac{1}{\hbar}(E_n - E_m)$. In fact, all frequencies in a spectrum can be expressed by quantities $\nu_{nm} = T_n - T_m$ where $n, m \in \mathbb{N}$. This fact was discovered by Ritz. Bohr used Planck's law to identify these T_n terms with energy via $E_n = hT_n$. Manipulations of these quantities were worked

out by Heisenberg and their relation to matrix calculus, already known at the time, was described by Born and Jordan. In some sense, Heisenberg discovered matrix calculus via physics. Born was the one who realized the canonical commutation relation

$$px - xp = \frac{h}{2\pi i} \tag{17}$$

between position x and momentum p.

Almost simultaneously, Dirac also described such manipulations and introduce the concept of a q-number (q for quantum), which basically corresponds to matrices. He did this also in 1925 (at approximately the age of 23!). Dirac later described the relationship between Poisson bracket of classical mechanics and the commutator of operators in quantum mechanics.

Schrödinger developed his ideas based on the concept of wave-particle duality in 1926, a few months after Heisenberg and Dirac but independently. In it, he introduced the concept of a wavefunction along with an ordinary differential equation which determines the future of this wavefunction. This differential equation is currently known as the Schrödinger equation and his theory at the time was called wave mechanics. Although the wavefunction is not a physically observable quantity, it is useful in describing observables. In fact, it was Schrödinger who declared the correspondence between linear operators and physical observables. Schrödinger was also responsible for showing that his wave mechanics is equivalent to Heisenberg's (and Dirac's) matrix mechanics.

The wavefunction gives the feeling that the notions of particles and waves are sort of blurred. On the one hand, the wavefunction occasionally has the form of an oscillatory function on space and on the other hand, it has a particle aspect to it coming from an interpretation of the wavefunction. This interpretation was supplied by Born. In the cases first studied, the wavefunction was a function on space, written as $\psi(x)$. Its interpretation was that the quantity $|\psi(x)|^2$ is the probability density of a particle to be located at the point x. In general, ψ could be a function of position, momentum, or energy. In that case, the probability is similarly described in terms of the *modulus squared* of ψ.

All this development lead to the following postulates. As a first run, we will be slightly imprecise, work through some examples to get a better feel for the physics, make precise certain mathematical nuances typically ignored by the physics community, and then come back to make more precise statements. This alone will take some time. Before the mathematician gets annoyed, let me make a personal remark. When a mathematician has a vague notion of what they want to do, but a precise statement or goal is not yet realized, and all the mathematician has is a bunch of examples and computations, then one must work a bit to achieve the statement of a theorem or something. And in the process of proving such a theorem, if the initial hypotheses were not enough, or too strong, then they are modified to give a better and more precise statement. This will be the approach taken here. Such an approach is probably more frustrating, but also more historical. Plus, it is the way physicists are taught. Anyway, we'll discuss the postulates and make remarks as we go on.

Postulate 1.1. *The physical states of a quantum system are in one to one correspondence with elements ψ in $\mathbb{P}\mathcal{H}$, where $\mathcal{H}$ is a complex separable Hilbert space and $\mathbb{P}\mathcal{H}$ is its projective space.*

In practice, one works with representative elements in the Hilbert space, instead of the projective space, so that the techniques of linear algebra and functional analysis can be used. The Hilbert space depends on the system one is describing. A physicist will not always say what Hilbert space they work with explicitly. Instead they will supply a set of commutation relations. These specify an algebra, and the Hilbert space is understood to be an irreducible representation of this algebra but a particular representation is not always chosen. Depending on the algebra, there are classification results that give unique (up to unitary equivalence) representations. We will discuss these in detail later.

At first, one might object to such an abstract definition of a state. To those thoughts, let me emphasize that a state, perhaps naively, should be described by some collection of numbers. Indeed, we will see that a state is completely described by precisely a collection of several types of numbers. These numbers turn out to correspond to vectors in Hilbert spaces (they correspond to eigenvalues of certain self-adjoint operators) and they uniquely specify a state in the Hilbert space (up to multiplication by a nonzero multiple of $\mathbb{C}$). In the following lecture, we will go over several important examples that illustrate this clearly.

Postulate 1.2. *To each real-valued function f on the classical phase space, there is a corresponding self-adjoint operator $\hat{f}$ on the quantum Hilbert space. $\hat{f}$ is called a quantum observable.*

Before we define adjoints, let me make two remarks.

First, as it stands, this statement is somewhat vacuous. One would like the operator $\hat{f}$ to satisfy some sort of property. For certain functions f and g on the classical phase space, $\hat{f}$ and $\hat{g}$ satisfy

$$\widehat{\{f,g\}} = -\frac{2\pi i}{h}[\hat{f},\hat{g}]. \tag{18}$$

There is no unique way to do this in general for all classical observables (this is known as the quantization ambiguity or also the Groene-van Hove Theorem). However, there is a unique way (up to unitary equivalence) to turn *some* classical observables into quantum operators, and in practice this is what one cares about in many situations. We will explore the general case in much detail when we talk about the actual process of quantization more precisely.

Second, the operator $\hat{f}$ might not be defined on the entire Hilbert space. This is more of a technical issue. We will talk a lot more about this in a future lecture (and in the examples in the next section).

To make sense of this postulate, we should review adjoints and Riesz' theorem.

Let $\mathcal{H}$ be a complex Hilbert space. Denote the inner product by $\langle \cdot\ ,\ \cdot \rangle$. Let A be a bounded operator on $\mathcal{H}$ (this means there exists a positive number C such that $\|A\psi\| \leq C\|\psi\|$ for all $\psi \in \mathcal{H}$). Then the adjoint $A^\dagger$ of A is a bounded operator on $\mathcal{H}$ such that

$$\langle \phi, A\psi \rangle = \langle A^\dagger \phi, \psi \rangle \quad \text{ for all } \phi, \psi \in \mathcal{H}. \tag{19}$$

Its existence follows from Riesz' theorem which states the following.

Theorem 1.3. *Using the same notation as in the previous paragraph, if $\xi : \mathcal{H} \to \mathbb{C}$ is a bounded linear functional, then there exists a unique $\phi_\xi \in \mathcal{H}$ such that*

$$\xi(\psi) = \langle \phi_\xi, \psi \rangle \quad \text{for all } \psi \in \mathcal{H}. \tag{20}$$

The reason the existence of $A^\dagger$ follows from this is because

$$\xi : \psi \mapsto \langle \phi, A\psi \rangle \tag{21}$$

is a bounded linear functional for any fixed $\phi \in \mathcal{H}$. Thus we define $A^\dagger \phi := \phi_\xi$.

With this, a bounded operator A is said to be self-adjoint if $A^\dagger = A$. A more technical description is needed for unbounded operators, which we will address later.

Postulate 1.4. *The expectation value of measuring an observable $\hat{f}$ in the quantum state ψ is given by $\langle \hat{f} \rangle_\psi := \langle \psi, \hat{f}\psi \rangle$.*

This postulate stresses that one does not get a precise measurement for an observable in general. Instead, what one gets is merely an *expectation value*. To illustrate what's happening, suppose that A is a self-adjoint operator *with* an orthonormal basis of eigenvectors (we will address the issue of the existence of eigenvectors in another lecture—this is a vert interesting question by the way where spectral theory and bundles are involved) denoted by $\{e_n\}_{n\in\mathbb{N} \text{ or } n\in\mathbb{Z}_N}$. Denote the corresponding eigenvalues by λ_n. Then any state ψ can be written uniquely as

$$\psi = \sum_n c_n e_n, \qquad \text{where} \quad c_n \in \mathbb{C} \tag{22}$$

Since ψ should be normalized to unity by the first postulate, we must have $\sum_n |c_n|^2 = 1$. Then the expectation value of the observable A for the state ψ is given by[1]

$$\langle \psi, A\psi \rangle = \left\langle \sum_n c_n^* e_n, \sum_m \lambda_m c_m e_m \right\rangle = \sum_{n,m} c_n^* \lambda_m c_m \langle e_n, e_m \rangle = \sum_n \lambda_n |c_n|^2. \tag{23}$$

What this means physically is that the probability of measuring the eigenvalue λ_n of the observable A for the state ψ is precisely $|c_n|^2$.

One does not always have eigenvectors for a given self-adjoint operator (for the physicist: the Dirac delta function is not an element in the Hilbert space). However, there is a notion of a generalized eigenspace and one obtains a probability measure on the spectrum of an operator describing the probability of a given measurement of a state ψ. This requires several definitions and the spectral theorem to describe fully and this will be done in a later lecture.

Postulate 1.5. *Let $\hat{f}$ be a quantum observable and let ψ be a quantum state. If upon measuring the observable $\hat{f}$ in the state ψ one obtains the eigenvalue λ, then a subsequent measurement made immediately afterwards will yield λ again.*

This is commonly referred to as the *collapse of the wave function*. It suggests that the observer has an irreversible effect on the state by act of measurement. Note that this is not at all the case in classical physics, which is deterministic and is described independent of an observer. In some sense, this brings up the question as to what exactly is measurement. I'd say this issue is still not completely clear at present.

[1] It is physics convention to conjugate the vector on the left in the inner product.

Postulate 1.6. *Let H denote the classical Hamiltonian operator for the classical system. The time evolution of a quantum state ψ is given by the Schrödinger equation*

$$\frac{d\psi}{dt} = -\frac{2\pi i}{h}\hat{H}\psi. \tag{24}$$

Postulate 1.3 says that all we can really obtain from a state is its expectation value. On the other hand, this postulate tells us that some information is completely deterministic. But remember, ψ is not an observable. The above postulate is also known as the Schrödinger representation. One can also formulate it in terms of operators by leaving elements in the Hilbert space fixed but considering the time evolution to take place for the operators. In this case, the time evolution is described by the Heisenberg equation of motion

$$\frac{d\hat{f}}{dt} = \frac{2\pi i}{h}[\hat{H}, \hat{f}]. \tag{25}$$

A personal comment is in order here since I cannot resist it. There is currently no agreed upon consensus how important classical physics is in the description of quantum mechanics. It seems that a classical theory is necessary in order to make sense of the quantum theory. It is my belief that there should exist a framework of quantum mechanics that makes no reference to classical mechanics. Furthermore, an understanding of this framework might be well-suited for a theory of quantum gravity [1]. Such a theory perhaps should have a prescription for how, in a certain range of parameters, give rise to classical physics. My speculation is that both classical physics and quantum physics are manifestations of some more general structure.

1.5. **Position and momentum.** The basic classical observables are position, momentum, and energy. Technically, in the end, position is really the most direct observable since we read dials, observe effects, and so on which take place at certain positions. Therefore, we will now supply the main Hilbert space which motivated quantum mechanics.

The Hilbert space associated to a classical particle of mass m moving in $\mathbb{R}^1$ is $L^2(\mathbb{R})$. The position operator $\hat{X}$ is defined by

$$(\hat{X}\psi)(x) = x\psi(x) \tag{26}$$

and the momentum operator P is defined by

$$(\hat{P}\psi)(x) = \frac{h}{2\pi i}\frac{d\psi(x)}{dx}. \tag{27}$$

One can easily check the commutation relation

$$\hat{P}\hat{X} - \hat{X}\hat{P} = \frac{h}{2\pi i}\mathrm{id}_{\mathcal{H}}. \tag{28}$$

Note that these equalities technically make sense only on their domain of definitions, but we will ignore this issue for now. $\hat{X}$ and $\hat{P}$ are examples of *unbounded operators*, something we will precisely define later. Clearly, there are nontrivial domains for all of these operators, so we should be satisfied by using the terminology "nice" to describe vectors where all of the statements we make hold. In fact, we have

$$\langle \phi, \hat{X}\psi\rangle = \langle \hat{X}\phi, \psi\rangle \tag{29}$$

and

$$\langle \phi, \hat{P}\psi \rangle = \langle \hat{P}\phi, \psi \rangle \tag{30}$$

for nice $\phi, \psi \in \mathcal{H}$.

2. Second lecture: the spectral theorem for bounded operators

As before, in the following, let $\mathcal{H}$ denote a complex separable Hilbert space. In the first lecture, we mentioned the importance of self-adjoint operators in quantum theory. Self-adjoint operators on $\mathcal{H}$ represent physical observables. We must set up some terminology that will be useful, namely bounded operators and adjoints. Then we can discuss self-adjoint operators along with several examples. From the examples, we motivate the definition of the spectrum of a self-adjoint operator and discuss its properties, most notably the spectral theorem.

2.1. **Bounded operators, examples, and adjoints.**

Definition 2.1. *A* bounded operator *from one Hilbert space $\mathcal{H}$ to another $\mathcal{H}'$ is a linear operator $L : \mathcal{H} \to \mathcal{H}'$ such that*

$$\sup_{\psi \in \mathcal{H}\setminus\{0\}} \left\{ \frac{\|L\psi\|}{\|\psi\|} \right\} < \infty. \tag{31}$$

If L is bounded, then

$$\|L\| := \sup_{\psi \in \mathcal{H}\setminus\{0\}} \left\{ \frac{\|L\psi\|}{\|\psi\|} \right\} \tag{32}$$

is called the norm *of the operator L.*

Remark 2.2. *Equivalently, the norm of L can be defined as*

$$\|L\| = \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \{\|L\psi\|\} \tag{33}$$

Example 2.3. *By the previous remark, in the case where $\mathcal{H} = \mathcal{H}'$ are finite-dimensional vector spaces, one can see that the norm of an operator coincides with the magnitude of the maximal eigenvalue of L. Thus, all linear operators between finite-dimensional Hilbert spaces are bounded.*

Example 2.4. *Let $\mathcal{H} = L^2([0,1])$. Recall that the position operator $\hat{X}$ is defined by*

$$(\hat{X}\psi)(x) := x\psi(x), \qquad \text{for all } \psi \in L^2([0,1]) \text{ and } x \in [0,1]. \tag{34}$$

$\hat{X}$ is bounded because

$$\|\hat{X}\psi\|^2 = \int_0^1 x^2 |\psi(x)|^2 \leq \int_0^1 |\psi(x)|^2 = 1 \tag{35}$$

for $\psi \in L^2([0,1])$ with $\|\psi\| = 1$.

Definition 2.5. *Let $A : \mathcal{H} \to \mathcal{H}'$ be a bounded operator. A* left adjoint to *A is a bounded operator $A^L : \mathcal{H}' \to \mathcal{H}$ such that*

$$\langle A^L\phi, \psi\rangle = \langle \phi, A\psi\rangle' \tag{36}$$

for all $\phi \in \mathcal{H}'$ and $\psi \in \mathcal{H}$. A right adjoint to *A is a bounded operator $A^R : \mathcal{H}' \to \mathcal{H}$ such that*

$$\langle \psi, A^R\phi\rangle = \langle A\psi, \phi\rangle' \tag{37}$$

for all $\phi \in \mathcal{H}'$ and $\psi \in \mathcal{H}$.

Because the inner product for a Hilbert space is conjugate symmetric, a left adjoint is a right adjoint and vice versa. Therefore, we simply call a left (and therefore right) adjoint of A as an *adjoint* of A, and we denote it by $A^\dagger$. We will discuss the existence and uniqueness of an adjoint in a moment, but first a simple example.

Example 2.6. *Let $\mathcal{H} = \mathbb{C}^n$ with the standard inner product. Then a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ can be written as a matrix with components A_{ij}. The adjoint $A^\dagger$ of A is given by the matrix with components A^*_{ji} where * denotes the complex conjugate. In this case, the adjoint exists and is unique.*

It is not an obvious statement, but adjoints to certain operators exist and are unique.

Theorem 2.7. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then $A^\dagger$ exists and is unique.*

We need an important Lemma, which we will not prove, for this theorem.

Theorem 2.8 (Riesz)**.** *Let $\xi : \mathcal{H} \to \mathbb{C}$ be a bounded operator. Then there exists a unique $\phi_\xi \in \mathcal{H}$ such that*

$$\xi(\psi) = \langle \phi_\xi, \psi \rangle \quad \text{for all } \psi \in \mathcal{H}. \tag{38}$$

We now prove the existence and uniqueness of adjoints.

Proof. Let $\phi \in \mathcal{H}$. Define $\xi : \mathcal{H} \to \mathbb{C}$ by

$$\xi : \psi \mapsto \langle \phi, A\psi \rangle. \tag{39}$$

This map is linear since A is linear and $\langle \cdot\,, \cdot \rangle$ is linear in the right coordinate. ξ is bounded because for all $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, we have

$$|\langle \phi, A\psi \rangle| \leq \|\phi\| \|A\| \|\psi\| = \|\phi\| \|A\| < \infty. \tag{40}$$

By Riesz' theorem, there exists a unique $\phi_\xi \in \mathcal{H}$ such that

$$\xi(\psi) = \langle \phi_\xi, \psi \rangle \quad \text{for all } \psi \in \mathcal{H}. \tag{41}$$

We define $A^\dagger$ by

$$A^\dagger \phi := \phi_\xi. \tag{42}$$

To show that $A^\dagger$ is linear, consider fixed $\phi_1, \phi_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ along with the following listen of equalities where we use the existence and uniqueness result just proved several times

$$\langle A^\dagger(\phi_1 + \lambda\phi_2), \psi \rangle = \langle \phi_1 + \lambda\phi_2, A\psi \rangle \tag{43}$$

$$= \langle \phi_1, A\psi \rangle + \lambda^* \langle \phi_2, A\psi \rangle \tag{44}$$

$$= \langle A^\dagger\phi_1, \psi \rangle + \lambda^* \langle A^\dagger\phi_2, \psi \rangle \tag{45}$$

$$= \langle A^\dagger\phi_1, \psi \rangle + \langle \lambda A^\dagger\phi_2, \psi \rangle \tag{46}$$

$$= \langle A^\dagger\phi_1 + \lambda A^\dagger\phi_2, \psi \rangle \tag{47}$$

for all ψ. Finally by uniqueness of the adjoint

$$A^\dagger(\phi_1 + \lambda\phi_2) = A^\dagger\phi_1 + \lambda A^\dagger\phi_2. \tag{48}$$

[Show $A^\dagger$ is a bounded] □

Definition 2.9. *A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is said to be* self-adjoint *if $A^\dagger = A$.*

Example 2.10. *The position operator $\hat{X}$ on $L^2([0,1])$ defined earlier is self-adjoint. This is because*

$$\langle \phi, \hat{X}\psi \rangle = \int_0^1 \phi(x)^* x\psi(x)dx = \int_0^1 (x\phi(x))^*\psi(x)dx = \langle \hat{X}\phi, \psi \rangle \tag{49}$$

for all $\psi, \phi \in L^2([0,1])$.

2.2. **Eigenvalues versus the spectrum.** Notice that in the previous example, the position operator does not have any eigenvectors! This should sound strange at first because in finite dimensions, we have the well-known result that any self-adjoint operator A on an n-dimensional Hilbert space has n eigenvectors with corresponding eigenvalues being real. This statement happens to be false in infinite dimensions. The physicist might be confused if they just listen to what their professors tell them. They might object and say that an eigenvector ψ to the equation

$$(\hat{X}\psi)(x) = \lambda\psi(x) \tag{50}$$

for a real number $\lambda \in [0,1]$ is the "function" $\psi(x) = \delta(x - \lambda)$ called the Dirac delta function concentrated at λ. There's nothing wrong with defining a measurable function to be zero almost everywhere and infinity at some points. However, that's not what the Dirac delta function is. Furthermore, it is not square-integrable. We will discuss the issue of eigen*vectors* more when we talk about direct integrals, but for now we illustrate an important point regarding the non-existence of eigen*values* first. Although there are no eigenvalues for the position operator, there is something called a *spectrum*. The spectrum of a self-adjoint operator, i.e. observable, corresponds to the values one can obtain upon measuring the observable in a particular state. Furthermore, we'd like to be able to associate probabilities of measuring such values in a particular range. The spectral theorem will let us do this.

Definition 2.11. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. The* spectrum *of A, denoted by $\sigma(A)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ does* not *have a bounded linear inverse.*

Why is this a good definition? We would like eigenvalues corresponding to non-zero eigenvectors of an operator to be in the spectrum, and clearly they are because if λ is an eigenvalue of A with corresponding eigenvector ψ, then $A - \lambda I$ cannot be invertible because $(A - \lambda I)\psi = A\psi - \lambda\psi = \lambda\psi - \lambda\psi = 0$ so that $A - \lambda I$ is not injective so it cannot be invertible. We now list several important properties for the spectrum of *bounded* linear operators as well as for self-adjoint operators.

Theorem 2.12. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator.*

i) Then the spectrum $\sigma(A)$ is a closed, bounded, and non-empty subspace of $\mathbb{C}$.
ii) If A is self-adjoint, then $\sigma(A) \subset \mathbb{R} \subset \mathbb{C}$.
iii) If A is self-adjoint, then $\lambda \in \mathbb{R}$ is in $\sigma(A)$ if and only if there exists a sequence ψ_n of nonzero vectors in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \frac{\|A\psi_n - \lambda\psi_n\|}{\|\psi_n\|} = 0 \tag{51}$$

The second statement is the analogue of the statement that self-adjoint operators in finite-dimensional Hilbert spaces have real eigenvalues. The third statement gives the feeling that for elements in the spectrum, there is a sequence of vectors that seem

to go towards the eigenvector, were one to exist. Since Hilbert spaces are complete, one can immediately check that the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ cannot be a Cauchy sequence unless λ is actually an eigenvector. As an example, one can consider a sequence of Gaussian functions concentrated at a fixed λ with decreasing width and increasing height for the position operator $\hat{X}$ on $L^2([0,1])$ defined in earlier examples. The limit is a distribution, the Dirac delta function, and is not in $L^2([0,1])$. This is a great segue to the following example.

Example 2.13. *Let $\hat{X}$ be the position operator on $L^2([0,1])$. Then $\sigma(\hat{X}) = [0,1]$.*

Proof. See Hall Example 7.9. □

2.3. **Measurements and the spectral theorem.** Now that we have the values that an observable, say A, can take, we want to associate to every "nice" subset $E \subset \sigma(A)$ a probability that a state will be found in such a range and we'd also like to find the subspace V_E in $\mathcal{H}$ corresponding to the states whose spectrum lies in the range E. Furthermore, by another postulate of quantum mechanics, we want the associated projection operator P_E since after a measurement in the range E, the state will necessarily be projected into the subspace V_E. First we discuss what "nice" means for subsets $E \subset \sigma(A)$. In order to do this rigorously, we need to introduce basic concepts from measure theory, which I expect is just a review. For those unfamiliar, measure theory gives us a rigorous way to express what we mean by length, area, volume, and so on. Furthermore, it gives us a way of doing analysis in more general settings.

Definition 2.14. *A* measurable space *is a set Y equipped with a collection Σ, called a sigma-algebra, of subsets of Y satisfying the following properties.*

i) $Y \in \Sigma$.
ii) Let C be any countable set. Then for any $E : C \to \Sigma$ the union $\cup_{c\in C} E(c) \in \Sigma$.
iii) If $E \in \Sigma$ then $E^c \in \Sigma$, where $E^c := \{y \in Y \mid y \notin E\}$ is the complement of E.

Example 2.15. *Let (Y,τ) be a topological space. The sigma-algebra generated by τ (the smallest sigma algebra containing τ) is called the Borel algebra and is denoted by $B(\tau)$. An element $E \in B(\tau)$ is called a Borel set.*

Definition 2.16. *Let (Y,Σ) and (Z,Ω) be two measure spaces. A function $f : Y \to Z$ is said to be* measurable *if $f^*(E) \in \Sigma$ for all $E \in \Omega$. Here f^* denotes the pullback of sets, namely $f^*(E) = \{y \in Y \mid f(y) \in E\}$.*

Example 2.17. *Let Y and Z be topological spaces and $f : Y \to Z$ a continuous map. Then f is Borel measurable. The converse need not be true, namely not every Borel measurable function is continuous.*

Definition 2.18. *Let (Y,Σ) be a measurable space. A positive measure on (Y,Σ) is a function $\mu : \Sigma \to [0,\infty]$ satisfying the following properties*

i) $\mu(E) \geq 0$ *for all* $E \in \Sigma$.
ii) $\mu(\varnothing) = 0$.
iii) Let C be a countable set. If $E : C \to \Sigma$ satisfies $E(c) \cap E(c') = \varnothing$ for any pair of two different $c, c' \in C$ then

$$\mu\left(\bigcup_{c\in C} E(c) \in \Sigma\right) = \sum_{c\in C} \mu(E(c)). \tag{52}$$

A measure tells us precisely how to assign volumes to measurable regions. What we want for quantum mechanics is a probability measure on $\sigma(A)$.

Definition 2.19. *Let (Y, Σ) be a measurable space. A* probability measure *on (Y, Σ) is a measure $\mu : \Sigma \to [0, \infty]$ with $\mu(Y) = 1$.*

In fact, we want something more in quantum mechanics. When we make a measurement, we project the original state onto a subspace. Depending on the measurement, the projection might be onto an eigenvector, but more general situations can occur as we've already mentioned for the position operator. If we make a measurement of an observable A in the range $E \subset \sigma(A)$, we want to not only associate a probability to such a range but also a projection operator P_E onto a closed subspace $V_E \subset \mathcal{H}$. To make sense of the equivalence between closed subspace of $\mathcal{H}$ and projection operators, we state but do not prove the following theorem.

Theorem 2.20. *There is a one-to-one correspondence between closed subspace $V \subset \mathcal{H}$ and self-adjoint bounded operators P such that $P^2 = P$. More precisely, for any closed subspace $V \subset \mathcal{H}$, there exists a unique bounded operator P_V, called the orthogonal projection onto V, such that $P_V^2|_V = I_V$ and $P_V|_{V^\perp} = 0$. Conversely, for any self-adjoint bounded operator P that satisfies $P^2 = P$ then P is an orthogonal projection onto $V_P := \text{range}(P)$.*

Finally, we can describe projection-valued measures. But first a notation. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. It is a fact that if $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a Banach space, and in fact a Banach algebra.

Definition 2.21. *Let (Y, Σ) be a measurable space. A projection-valued measure on (Y, Σ) is a function $\mu : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions.*

i) $\mu(E)$ *is an orthogonal projection for every* $E \in \Sigma$.

ii) $\mu(\varnothing) = 0$ *and* $\mu(Y) = I$.

iii) Let C be a countable set. If $E : C \to \Sigma$ satisfies $E(c) \cap E(c') = \varnothing$ for any pair of two different $c, c' \in C$ then

$$\mu\left(\bigcup_{c\in C} E(c)\right) = \sum_{c\in C} \mu(E(c)) \tag{53}$$

meaning that for all $\psi \in \mathcal{H}$

$$\mu\left(\bigcup_{c\in C} E(c)\right)\psi = \sum_{c\in C} \mu(E(c))\psi. \tag{54}$$

iv) For all $E_1, E_2 \in \Sigma$,

$$\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2). \tag{55}$$

Remark 2.22. *The reader might realize the similarity between a projection-valued measure and a probability measure. In particular, notice how the measure of the entire X is the identity operator, which has norm* 1.

One of the purposes of all these definitions is that we want to associate to an observable A a measure μ^A on $\sigma(A)$, the spectrum of A, which will tell us information about the expectation values of the observable A and certain powers A^m including the probability of measuring A in particular ranges. But for this, we have to set up a calculus for integrating over ranges $E \subset \sigma(A)$. First we give an example relating projection-valued measures to ordinary measures.

Example 2.23. *Let μ be a projection-valued measure on a measurable space (Y, Σ). Let $\psi \in \mathcal{H}$ be any vector. Then one can define a positive measure μ_ψ on (Y, Σ) by setting*

$$\mu_\psi(E) := \langle \psi, \mu(E)\psi \rangle \tag{56}$$

for all $E \in \Sigma$. In the special case where $\mathcal{H} = L^2([0,1])$, this says

$$\mu_\psi(E) := \int_0^1 \psi(x)^* (\mu(E)\psi)(x)\, dx \tag{57}$$

so that a projection-valued measure gives rise to integration.

This motivates the following definition and theorem which generalizes the previous example.

Theorem 2.24. *Let (Y, Σ) be a measurable space with a projection-valued measure μ. Then there exists a unique linear map*

$$L^\infty(Y) \xrightarrow{\int_Y \cdot \, d\mu} \mathcal{B}(\mathcal{H}) \tag{58}$$

$$f \longmapsto \int_Y f \, d\mu \tag{59}$$

from the space of bounded and measurable complex-valued functions on Y into bounded operators on $\mathcal{H}$ with the property that

$$\left\langle \psi, \left(\int_Y f \, d\mu \right) \psi \right\rangle = \int_Y f \, d\mu_\psi \tag{60}$$

for all $f : Y \to \mathbb{C}$ and $\psi \in \mathcal{H}$. The integral operator satisfies several more properties (see Hall Proposition 7.11).

Theorem 2.25 (The spectral theorem for bounded operators)**.** *For any self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ there exists a unique projection-valued measure μ^A on the Borel sigma algebra for $\sigma(A)$ such that*

$$\int_{\sigma(A)} \lambda \, d\mu^A(\lambda) = A. \tag{61}$$

There is a different version of the spectral theorem which we will discuss next time which associates a more natural measure to $\sigma(A)$ that does not come from the Borel sigma algebra from $\mathbb{R}$ and gives a sort of decomposition of the Hilbert space into a collection of Hilbert spaces living over the spectrum.

Theorem 2.26. *For any self-adjoint operator A on $\mathcal{H}$ and any unit vector $\psi \in \mathcal{H}$, there exists a unique probability measure μ_ψ^A on $\sigma(A)$ such that*

$$\int_{\sigma(A)} \lambda^m \, d\mu_\psi^A(\lambda) = \langle \psi, A^m \psi \rangle \tag{62}$$

for all $m \in \mathbb{N}$.

3. Third lecture: a geometric version of the spectral theorem for bounded operators

Today we will discuss another version of the spectral theorem that will enable us to describe self-adjoint bounded operators in terms of multiplication operators and give a meaning to the notion of distributions as being eigenvectors of certain multiplication operators but these eigenvectors will live in another Hilbert space. To do this, we will need to discuss families of Hilbert spaces that live over the spectrum of a bounded operator in terms of what are called *direct integrals*. We will recall some definitions to make this as self-contained as possible. As usual let $\mathcal{H}$ denote a complex separable Hilbert space.

3.1. **Position, momentum, and the Fourier transform.**

Definition 3.1. *Let $\mathcal{H} = L^2(Y, \mu)$ be the space of square-integrable measurable functions on a measure space (Y, μ). Let h be a bounded measurable real-valued function on X. Let M_h be the operator defined by*

$$M_h\psi := h\psi \tag{63}$$

for all $\psi \in L^2(Y, \mu)$. M_h is called a multiplication operator *on $L^2(Y, \mu)$.*

Example 3.2. *Recall the case $L^2([0, 1])$ which is the Hilbert space describing the possible states of a particle living on the interval $[0, 1]$. We considered the position operator $\hat{X}$ which was defined by*

$$(\hat{X}\psi)(x) := x\psi(x) \tag{64}$$

for all $x \in [0, 1]$ and all $\psi \in L^2(\mathbb{R})$. This is an example of a multiplication operator where the function associated to this operator is actually the identity.

Sometimes non-examples help as well.

Example 3.3. *Consider again the case $L^2([0, 1])$ but with the momentum operator $\hat{P}$ defined by*

$$(\hat{P}\psi)(x) := \frac{h}{2\pi i}\frac{d}{dx}\psi(x) \tag{65}$$

which poses a few issues because for one it is not actually defined on all of $L^2([0, 1])$ but only a dense subspace. $\hat{P}$ is an example, in this case, of what's called an unbounded operator*. We will discuss unbounded operators in the next lecture. Physically, this operator describes the momentum of a particle living on $[0, 1]$. Regardless, we can still consider it and note that it is clearly not a multiplication operator.*

The two examples above are related, although we will have to wait to give a precise formulation where we state the spectral theorem for unbounded operators. However, we can still give a flavor of what to expect in this general case. The second version of the spectral theorem that we will discuss says that any bounded self-adjoint operator on $\mathcal{H}$ is unitarily equivalent to a multiplication operator on a direct integral. Briefly, the direct integral in this version of the spectral theorem is the collection of all assignments of elements in a family of Hilbert spaces living over a measure space, which sometimes is actually the spectrum of that self-adjoint operator. The result will be that the position and momentum operators are precisely related by the spectral theorem in this way. If we ignore the technicality of

the direct integral and allow ourselves to think that Dirac delta distributions are elements of our Hilbert space then the spectral theorem says the following.

Recall that the Fourier transform operator $\mathcal{F}$ is defined on (a subset of) $L^2(\mathbb{R})$ by the following formula

$$(\mathcal{F}\psi)(p) := \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} e^{-2\pi i p x/h} \psi(x) dx \tag{66}$$

In Michael's talk [7], we discussed under what conditions does this formula make sense and produces a nice result. Since we're being vague, we will say that this works for "nice" ψ. Anyway, we find the following fact

$$(\mathcal{F}(\hat{P}\psi))(p) = p(\mathcal{F}\psi)(p) \tag{67}$$

by simply applying integration by parts. Thus, we've turned the derivative operator described by $\hat{P}$ into a multiplication operator. We will discuss this more generally later, and in this lecture specifically for bounded operators.

3.2. **Direct integrals.** But to do so, we have to introduce a notion that not only has importance in describing another version of the spectral theorem, but also has significant importance in a generalization of the Peter-Weyl theorem for locally compact Lie groups. This is the concept of the direct integral.

Definition 3.4. *Let (Y, Σ, μ) be a measure space. A* Hilbert family *over Y consists of a complex separable Hilbert space $\mathcal{H}_\lambda$ with associated inner product $\langle \cdot\ ,\ \cdot \rangle_\lambda$ for every $\lambda \in Y$. There is a natural function $\pi : \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \to Y$ called the* projection *of the Hilbert family.*

Definition 3.5. *A* section *of a Hilbert family $\pi : \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \to Y$ is a function $s : Y \to \bigcup_{\lambda \in Y} \mathcal{H}_\lambda$ such that $\pi \circ s = \mathrm{id}_Y$. Denote the set of sections of $\pi : \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \to Y$ by $\Gamma(\pi)$.*

The goal now is to define an inner product on the set of sections. Given two sections $s, t \in \Gamma(\pi)$, one candidate for the inner product would be

$$\langle s, t \rangle = \int_Y \langle s(\lambda), t(\lambda) \rangle_\lambda d\mu(\lambda). \tag{68}$$

The only problem is that the function

$$Y \to \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \tag{69}$$

$$\lambda \mapsto \langle s(\lambda), t(\lambda) \rangle_\lambda \tag{70}$$

must be measurable *for all sections s and t*. Otherwise, one could not take the integral of such a function with respect to the measure μ. The problem is then to define the notion of measurability for sections so that for any two measurable sections s and t, the above function will be measurable. There are several ways to implement this condition. One way is to demand the existence of a countable family of sections $\{e_j\}_{j \in J}$ such that

$$\langle e_j(\lambda), e_k(\lambda) \rangle_\lambda = \begin{cases} 0 & \text{if } j \neq k \\ 1 \text{ or } 0 & \text{if } j = k \end{cases} \tag{71}$$

and

$$\overline{\mathrm{span}(\{e_j(\lambda)\}_{j \in J})} = \mathcal{H}_\lambda \tag{72}$$

for all λ. With this, one can make the following definition of measurability of sections.

Definition 3.6. *Let $\pi : \bigcup_{\lambda\in Y} \mathcal{H}_\lambda \to Y$ be a Hilbert family over a measure space (Y, Σ, μ) and let $\{e_j\}_{j\in J}$ be a countable family of sections satisfying the above conditions. Then a section $s : Y \to \bigcup_{\lambda\in Y} \mathcal{H}_\lambda$ is* measurable *if the function*

$$\lambda \mapsto \langle e_j(\lambda), s(\lambda)\rangle_\lambda \tag{73}$$

is measurable for all $j \in J$. The collection $\{e_j\}_{j\in J}$ is called a measurability structure *on the Hilbert family $\pi : \bigcup_{\lambda\in Y} \mathcal{H}_\lambda \to Y$. The set of all measurable sections of π will be denoted by $\Gamma_\mu(\pi)$.*

Lemma 3.7. *Let $\{e_j\}_{j\in J}$ be a measurability structure on the Hilbert family $\pi : \bigcup_{\lambda\in Y} \mathcal{H}_\lambda \to Y$ and let s and t be two measurable sections. Then the function*

$$\lambda \mapsto \langle s(\lambda), t(\lambda)\rangle_\lambda \tag{74}$$

is measurable.

Proof. □

Example 3.8. *Let $\mathcal{H}$ be a complex separable Hilbert space. Then $\pi : Y \times \mathcal{H} \to Y$ is a Hilbert family. It is called a* trivial *Hilbert family with fiber $\mathcal{H}$. A measurability structure clearly exists in this case. This is because there exists a countable orthonormal basis $\{e_j\}_{j\in J}$ for $\mathcal{H}$ by the separability condition on $\mathcal{H}$. Furthermore, there is a bijection $\Gamma_\mu(\pi) \cong \mathrm{Meas}(X, \mathcal{H})$, where the latter set is the set of measurable functions from X to $\mathcal{H}$. Here $\mathcal{H}$ is equipped with the Borel measure.*

Remark 3.9. *The importance of Hilbert families is that they generalize the previous example to more interesting situations where one considers functions that are only locally defined and take values in possibly different Hilbert spaces. The notion of locally defined objects in physics is crucial for understanding physics at a global scale. For instance, the Dirac monopole serves as a wonderful example of where the notion of a section is* crucial *to the understanding of what a wave function is. A grand scale version of this was put forward by Dan Freed and Constantin Teleman [4], who basically say that fields in physics are stacks (a generalization of the notion of a presheaf, which in turn generalizes vector families).*

Lemma 3.10. *Let $\{e_j\}_{j\in J}$ and $\{e'_{j'}\}_{j'\in J'}$ be two measurability structures on the Hilbert family $\pi : \bigcup_{\lambda\in Y} \mathcal{H}_\lambda \to Y$. Then a section s is measurable with respect to $\{e_j\}_{j\in J}$ if and only if it is measurable with respect to $\{e'_{j'}\}_{j'\in J'}$.*

Proof. □

Definition 3.11. *Let (Y, Σ, μ) be a measure space. Let $\pi : \bigcup_{\lambda\in Y} \mathcal{H}_\lambda \to Y$ be a Hilbert family. The set of equivalence classes of almost everywhere equal square-integrable sections, namely $s \in \Gamma_\mu(\pi)$ such that*

$$\|s\| := \sqrt{\langle s, s\rangle} = \left(\int_Y \langle s(\lambda), s(\lambda)\rangle_\lambda d\mu(\lambda)\right)^{1/2} < \infty, \tag{75}$$

is denoted by $\mathcal{L}^2_\mu(\pi)$.

The following is the crucial definition of this lecture and is also a theorem, which we will prove.

Lemma 3.12. *With (Y, Σ, μ) and $\pi : \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \to Y$ as above, the assignment*

$$(s,t) \mapsto \langle s, t \rangle = \int_Y \langle s(\lambda), t(\lambda) \rangle_\lambda d\mu(\lambda) \tag{76}$$

defines an inner product on $\mathcal{L}^2_\mu(\pi)$ and therefore gives $\mathcal{L}^2_\mu(\pi)$ the structure of a Hilbert space. This Hilbert space is called a direct integral *and is sometimes written as*

$$\mathcal{L}^2_\mu(\pi) \equiv \int_Y^\oplus \mathcal{H}_\lambda \, d\mu(\lambda). \tag{77}$$

Because two different measurability structures define the same set of measurable functions, it is useful to know that a measurability structure exists rather than specifying one. This is the motivation behind the following Lemma.

Lemma 3.13. *Let (Y, Σ, μ) and $\pi : \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \to Y$ be as above. Suppose that the function*

$$Y \to [0, \infty] \tag{78}$$

$$\lambda \mapsto \dim \mathcal{H}_\lambda \tag{79}$$

is measurable. Then there exists a measurability structure on the Hilbert family.

Proof. □

Example 3.14. *Let Y be a countable measurable set, Σ the set of all subsets of Y, and μ the counting measure on Y. Let $\{\mathcal{H}_\lambda : \lambda \in Y\}$ be a collection of Hilbert spaces. Then*

$$\int_Y^\oplus \mathcal{H}_\lambda \, d\mu(\lambda) = \bigoplus_{\lambda \in Y} \mathcal{H}_\lambda. \tag{80}$$

Notice that in the case of the previous example, $\mathcal{H}_{\lambda_0}$ embeds into $\int_Y^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$. This actually happens under the following conditions.

Proposition 3.15. *Let $\int_Y^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$ be a direct integral for a measure space (Y, Σ, μ) and Hilbert family $\pi : \bigcup_{\lambda \in Y} \mathcal{H}_\lambda \to Y$. If $\lambda_0 \in Y$ has measure $\mu(\lambda_0) > 0$, then the map*

$$\mathcal{H}_{\lambda_0} \to \int_Y^\oplus \mathcal{H}_\lambda \, d\mu(\lambda) \tag{81}$$

$$\psi \mapsto \left(\lambda \mapsto \begin{cases} \frac{\psi}{\sqrt{\mu(\lambda_0)}} & \text{if } \lambda = \lambda_0 \\ 0 & \text{otherwise} \end{cases} \right) \tag{82}$$

defines an isometric embedding of $\mathcal{H}_{\lambda_0}$ in $\int_Y^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$.

3.3. **The spectral theorem via direct integrals.**

Theorem 3.16. *Let A be a bounded self-adjoint operator on $\mathcal{H}$. Then there exists a σ-finite measure μ on $\sigma(A)$, a direct integral $\int_{\sigma(A)}^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$, and a unitary map $U : \mathcal{H} \to \int_{\sigma(A)}^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$ such that*

$$[UAU^{-1}(s)](\lambda) = \lambda s(\lambda) \tag{83}$$

for all sections $s \in \int_{\sigma(A)}^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$.

Here's how I understand the idea of a direct integral associated to a self-adjoint operator A. First let's suppose that the spectrum $\sigma(A)$ solely contains eigenvalues and is discrete and countable. Let λ_0 be such an eigenvalue. Then $\mathcal{H}_{\lambda_0}$ is the Hilbert subspace of $\bigoplus_{\lambda\in Y}\mathcal{H}_\lambda$ that contains all vectors with eigenvalue λ_0. In physics parlance, this is called *degeneracy*. A great example of such an operator is the Hamiltonian (energy) for a two-dimensional simple harmonic oscillator. Another example is a system with spherical symmetry. Then the square of the angular momentum operator will have degeneracy. In general, the spectrum $\sigma(A)$ of A need not contain any eigenvalues. The description above still applies in a more general sense. One typically thinks of the Hilbert family as containing the information about degeneracy for generalized eigenvectors. This will all be made more precise with explicit examples after the spectral theorem for unbounded operators is discussed.

The above theorem can be modified slightly so that one works over a measure space (not necessarily the spectrum of A) with a Hilbert family over it where the dimensions are all one. In other words, the sections are simply ordinary square-integrable functions over this measure space. The cost of this simplicity is that the multiplication operator becomes slightly more complicated (rather than just the identity function).

Theorem 3.17. *Let A be a bounded self-adjoint operator on $\mathcal{H}$. Then there exists a σ-finite measure space (Y,Σ,μ), a bounded, measurable, and real-valued function $h: Y\to\mathbb{R}$, and a unitary operator $U:\mathcal{H}\to L^2(Y)$ such that*

$$[UAU^{-1}(\psi)](\lambda) = h(\lambda)\psi(\lambda) \tag{84}$$

for all $\psi\in L^2(Y)$ and $\lambda\in Y$.

There are two theorems that give sort of uniqueness results. But first a definition.

Definition 3.18. *Let μ and ν be two measures on (Y,Σ). μ is said to be* absolutely continuous *with respect to ν if for all $E\in\Sigma$ with $\nu(E)=0$ then $\mu(E)=0$. When this happens, we write $\mu\leq\nu$. μ and ν are said to be* equivalent *or* mutually absolutely continuous *if $\mu\leq\nu$ and $\nu\leq\mu$.*

Theorem 3.19. *Let A be a bounded self-adjoint operator on $\mathcal{H}$. Let $\mu^{(1)}$ and $\mu^{(2)}$ be two σ-finite measures on $\sigma(A)$ with two associated direct integrals $\int_{\sigma(A)}^{\oplus}\mathcal{H}_\lambda^{(1)}\,d\mu^{(1)}(\lambda)$ and $\int_{\sigma(A)}^{\oplus}\mathcal{H}_\lambda^{(2)}\,d\mu^{(2)}(\lambda)$. If furthermore (one can always make this happen without changing the direct integral) $\dim\mathcal{H}_\lambda^{(j)}>0$ for almost all λ with respect to $\mu^{(j)}$ for both $j=1,2$, then $\mu^{(1)}$ and $\mu^{(2)}$ are mutually absolutely continuous and*

$$\dim\mathcal{H}_\lambda^{(1)} = \dim\mathcal{H}_\lambda^{(2)} \tag{85}$$

for almost all λ with respect to $\mu^{(j)}$ for both $j=1,2$.

Although you can't hear the shape of a drum [5], the following theorem gives a result for how to determine an operator up to unitary equivalence from its spectrum and additional data.

Definition 3.20. *Let $A^{(1)}\in\mathcal{B}(\mathcal{H}_1)$ and $A^{(2)}\in\mathcal{B}(\mathcal{H}_2)$. Then $A^{(1)}$ is unitarily equivalent to $A^{(2)}$ if there exists a unitary operator $U:\mathcal{H}_1\to\mathcal{H}_2$ such that*

$$A^{(2)} = UA^{(1)}U^{-1}. \tag{86}$$

Theorem 3.21. *Let $A^{(1)} \in \mathcal{B}(\mathcal{H}^{(1)})$ and $A^{(2)} \in \mathcal{B}(\mathcal{H}^{(2)})$ be bounded operators on separable Hilbert spaces. Let $(\sigma(A^{(1)}), \Sigma^{(1)}, \mu^{(1)})$ and $(\sigma(A^{(2)}), \Sigma^{(2)}, \mu^{(2)})$ be two σ-finite measure spaces with associated direct integrals $\int^{\oplus}_{\sigma(A^{(1)})} \mathcal{H}^{(1)}_\lambda \; d\mu^{(1)}(\lambda)$ and $\int^{\oplus}_{\sigma(A^{(2)})} \mathcal{H}^{(2)}_\lambda \; d\mu^{(2)}(\lambda)$ satisfying $\dim \mathcal{H}^{(j)}_\lambda > 0$ for almost all λ with respect to $\mu^{(j)}$ for both $j = 1, 2$. Then $A^{(1)}$ and $A^{(2)}$ are unitarily equivalent if and only if the following conditions are satisfied.*

i) The spectra of both $A^{(1)}$ and $A^{(2)}$ are equal, $\sigma(A^{(1)}) = \sigma(A^{(2)})$.

ii) The measures $\mu^{(1)}$ and $\mu^{(2)}$ are mutually absolutely continuous.

iii) The two functions

$$\lambda \mapsto \dim \mathcal{H}^{(1)}_\lambda \qquad \text{and} \qquad \lambda \mapsto \dim \mathcal{H}^{(2)}_\lambda \tag{87}$$

agree almost everywhere.

4. Fourth lecture: the spectral theorem for unbounded operators

In this lecture, we will discuss unbounded operators, the spectrum of an unbounded operator, and the spectral theorem in its two forms. One version gives us a way to discuss functions of unbounded operators while the other version describes generalized eigenspaces for unbounded operators in terms of a direct integral. Most of this lecture is a repetition of what was done earlier for bounded operators. Few examples will be given in this lecture since many of them were already mentioned earlier. Finally, we will state Stone's theorem which is a wonderful result that allows us to forget about issues of domains of operators provided that we work with a different set of operators.

4.1. Unbounded operators.

We've encountered several unbounded operators that we'd like to consider as physical observables/self-adjoint operators. However, the following fact shows that it is impossible to be simultaneously self-adjoint and *not* bounded.

Proposition 4.1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. If A satisfies*

$$\langle A\psi, \phi\rangle = \langle \psi, A\phi\rangle \tag{88}$$

for all $\phi, \psi \in \mathcal{H}$, then A is bounded.

Read in its contrapositive, if A is not bounded, A cannot be self-adjoint.

Proof. The proof involves a bit of analysis. See Corollary 9.9 of Hall. □

Unbounded operators are typical in quantum theory. Therefore, we should modify what we mean by self-adjointness to allow for such a notion for unbounded operators. First we recall what an unbounded operator is.

Definition 4.2. *An* unbounded operator *A on a Hilbert space $\mathcal{H}$ is a linear map on some dense domain $\mathrm{dom}(A) \subset \mathcal{H}$ to $\mathcal{H}$:*

$$A : \mathrm{dom}(A) \to \mathcal{H}. \tag{89}$$

Remark 4.3. *One awkward issue with unbounded operators is that combinations of such operators might not result in unbounded operators, namely, they might not be densely defined. For instance, if A and B are unbounded operators on $\mathcal{H}$, then $A+B$ is a linear map defined on $\mathrm{dom}(A+B) = \mathrm{dom}(A) \cap \mathrm{dom}(B)$. Even if each separate domain is dense, the intersection need not be. However, there are several different types of sufficient conditions that guarantee certain nice properties (for instance, see Proposition 9.13 and Theorem 9.37 of Hall). The product of two operators will be discussed when we get to the spectral theorem.*

Notice that an unbounded operator A is not even bounded on its domain! Indeed, if it was, it could be extended uniquely to a bounded operator on all of $\mathcal{H}$. This follows from the Bounded Linear Transformation Theorem, which is reproduced here for convenience.

Theorem 4.4. *Let V be a normed vector space, W a Banach space, and A a bounded linear map defined on a dense domain $\mathrm{dom}(A)$ of V to W, i.e.*

$$A : \mathrm{dom}(A) \to W. \tag{90}$$

Then there exists a unique bounded linear map $\tilde{A} : V \to W$ such that $\tilde{A}|_{\mathrm{dom}(A)} = A$.

This theorem allows us to make a definition for the adjoint of an *unbounded* operator.

Definition 4.5. *Let A be an unbounded operator on a Hilbert space $\mathcal{H}$ with domain* $\mathrm{dom}(A)$. *Let $\phi \in \mathcal{H}$ be a vector such that the linear functional*

$$\mathrm{dom}(A) \to \mathbb{C} \tag{91}$$

$$\psi \mapsto \langle \phi, A\psi \rangle \tag{92}$$

is bounded. Then by Riesz' theorem (Theorem 1.3), there exists a unique $\chi_\phi \in \mathcal{H}$ such that $\langle \phi, A \cdot \rangle = \langle \chi_\phi, \cdot \rangle$. Define the operator $A^\dagger$ on such ϕ by $A^\dagger \phi := \chi_\phi$. Then the domain of $A^\dagger$ is the set of all such ϕ for which Riesz' theorem can be applied

$$\mathrm{dom}(A^\dagger) := \{\phi : \langle \phi, A \cdot \rangle \text{ is bounded for all } \phi \in \mathrm{dom}(A)\}. \tag{93}$$

This defines/constructs the adjoint $A^\dagger$ of A.

Note that $A^\dagger$ is only *linear* on its domain. It need not be bounded! Furthermore, even if $\mathrm{dom}(A)$ is dense in $\mathcal{H}$, $\mathrm{dom}(A^\dagger)$ need not be.

Definition 4.6. *An unbounded operator A on $\mathcal{H}$ is* self-adjoint *if* $\mathrm{dom}(A^\dagger) = \mathrm{dom}(A)$ *and $A = A^\dagger$ on these domains.*

From now on, the term "self-adjoint operator" will be short for "self-adjoint unbounded operator." Now that we've modified what we mean by self-adjointness, we should also modify what we mean by the spectrum of an unbounded operator. Many of the theorems we've stated before for bounded operators remain true in the unbounded operator case.

Definition 4.7. *Let A be an unbounded operator on $\mathcal{H}$ with domain* $\mathrm{dom}(A)$. *The* spectrum of A, *denoted by $\sigma(A)$, is the set of all $\lambda \in \mathbb{C}$ such that there does* not *exist a bounded operator B with the following two properties.*

i) For all $\psi \in \mathcal{H}$, $B\psi \in \mathrm{dom}(A)$ and $(A - \lambda I)B\psi = \psi$.
ii) For all $\phi \in \mathrm{dom}(A)$, $B(A - \lambda I)\phi = \phi$.

Theorem 4.8. *Let $A : \mathcal{H} \to \mathcal{H}$ be an unbounded operator.*

i) Then the spectrum $\sigma(A)$ is a closed subspace of $\mathbb{C}$.
ii) If A is self-adjoint, then $\sigma(A) \subset \mathbb{R} \subset \mathbb{C}$.
iii) If A is self-adjoint, then $\lambda \in \mathbb{R}$ is in $\sigma(A)$ if and only if there exists a sequence ψ_n of nonzero vectors in $\mathrm{dom}(A)$ *such that*

$$\lim_{n\to\infty} \frac{\|A\psi_n - \lambda\psi_n\|}{\|\psi_n\|} = 0 \tag{94}$$

Note that the spectrum of an unbounded operator is not necessarily bounded (and therefore not necessarily compact). In fact, several more interesting facts are true but one uses the spectral theorem to prove these results. One of these is a strengthening of the previous statement. Namely, if A is self-adjoint and has a *bounded* spectrum, then it is necessarily bounded!

Another important fact can be stated with regards to the spectrum.

Definition 4.9. *Let A be an unbounded operator on $\mathcal{H}$ with domain* $\mathrm{dom}(A)$. *Then A is said to be* bounded below *by a constant $c \in \mathbb{R}$ if $\langle \psi, A\psi \rangle \geq c\|\psi\|^2$ for all $\psi \in \mathrm{dom}(A)$. If $c = 0$, then A is said to be* non-negative.

Proposition 4.10. *Let A be an unbounded operator on $\mathcal{H}$. If A is bounded below by c, then $\sigma(A) \subset [c, \infty)$.*

Furthermore, an interesting result of the spectral theorem is that if $\sigma(A) \subset [c, \infty)$, then A is bounded below by c.

4.2. **Some facts about physical operators in quantum mechanics.** Here we merely state some theorems about operators of interest in quantum mechanics and also some theorems that tell us when certain sums of unbounded operators are still unbounded. We also give an example of a situation where this fails.

Theorem 4.11. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and define the linear operator $V(\hat{X})$ on a domain $\mathrm{dom}(V(\hat{X}))$ in $L^2(\mathbb{R}^n)$ by*

$$(V(\hat{X})\psi)(x) := V(x)\psi(x) \tag{95}$$

for all ψ in the domain

$$\mathrm{dom}(V(\hat{X})) := \left\{\psi \in L^2(\mathbb{R}^n) \mid (x \mapsto V(x)\psi(x)) \in L^2(\mathbb{R}^n)\right\}. \tag{96}$$

Then $\mathrm{dom}(V(\hat{X}))$ is dense in $L^2(\mathbb{R}^n)$ and $V(\hat{X})$ is self-adjoint on this domain.

In particular, the position operator is self-adjoint on its respected domain. By Fourier transform, it also follows that the momentum operator is self-adjoint on its respected domain. One could explicitly prove that the momentum operator has a dense domain and that it is self-adjoint on this domain explicitly using a notion of derivatives for functions in $L^2(\mathbb{R}^n)$ but we won't describe this. Similarly, if $\hat{P}$ is the momentum operator, then $\hat{P}^2$ is also densely defined and self-adjoint on its domain. A difficult theorem to prove that combines the previous results is the following (one that most physicists take for granted).

Theorem 4.12. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a measurable function with $n \leq 3$. If V can be decomposed as $V = V_1 + V_2$ with both V_1 and V_2 real, measurable, and V_1 is in $L^2(\mathbb{R}^n)$ while V_2 is bounded, then the operator $\frac{\hat{P}^2}{2m} + V(\hat{X})$ is self-adjoint on $\mathrm{dom}(\hat{P}^2)$.*

The operator $\frac{\hat{P}^2}{2m} - \hat{X}^4$ is not self-adjoint (nor is it essentially self-adjoint) on $C_c^\infty(\mathbb{R})$, the set of smooth compactly supported functions on $\mathbb{R}$. We didn't define what essentially self-adjoint means but it is a slightly weaker notion than self-adjoint.

4.3. **The spectral theorem.** First, we will describe how to take a measurable function, for now on an arbitrary measurable space (Y, Σ), and with a projection-valued measure, obtain an unbounded operator. This is important because when we discuss operators in quantum mechanics, sometimes operators are plugged into the domain of a function such as $V(\hat{X})$ in our earlier theorem from today. Recall that for any projection-valued measure μ on (Y, Σ), and any state $\psi \in \mathcal{H}$, one can construct a non-negative, real-valued *ordinary* measure μ_ψ on Y by the formula $\mu_\psi(E) := \langle \psi, \mu(E)\psi \rangle$. The physical significance of this will be explained when we consider the special case of $Y = \sigma(A)$, the spectrum of a self-adjoint operator. The real number $\mu_\psi(E)$ will represent the probability of measuring a state ψ (provided $\|\psi\| = 1$) to have a particular value for the operator A in the subset $E \subset \sigma(A)$.

Proposition 4.13. *Let (Y, Σ) be a measurable space and μ a projection-valued measure on it with values in $\mathcal{B}(\mathcal{H})$. Let $f : Y \to \mathbb{C}$ be a measurable function. Then there exists a unique unbounded operator $\int_Y f d\mu$ on $\mathcal{H}$ with domain*

$$\operatorname{dom}\left(\int_Y f d\mu\right) := \left\{\psi \in \mathcal{H} \mid \int_Y |f(\lambda)|^2 d\mu_\psi(\lambda) < \infty\right\} \tag{97}$$

such that

$$\left\langle \psi, \left(\int_Y f d\mu\right)\psi \right\rangle = \int_Y f(\lambda) d\mu_\psi(\lambda) \tag{98}$$

for all $\psi \in \operatorname{dom}\left(\int_Y f d\mu\right)$. Furthermore,

$$\left\langle \left(\int_Y f d\mu\right)\psi, \left(\int_Y f d\mu\right)\psi \right\rangle = \int_Y |f(\lambda)|^2 d\mu_\psi(\lambda) \tag{99}$$

for all $\psi \in \operatorname{dom}\left(\int_Y f d\mu\right)$.

This last condition should explain why the domain of $\int_Y f d\mu$ is defined the way it is. In general, one wants $\left(\int_Y f d\mu\right)\psi$ to still be in $\mathcal{H}$, which means that

$$\left\langle \left(\int_Y f d\mu\right)\psi, \left(\int_Y f d\mu\right)\psi \right\rangle < \infty \tag{100}$$

should hold.

Proposition 4.14. *Using the assumptions from the previous theorems, if f is bounded, then $\operatorname{dom}\left(\int_Y f d\mu\right) = \mathcal{H}$ and $\int_Y f d\mu$ is a bounded operator. If f is real-valued, then $\int_Y f d\mu$ is self-adjoint on $\operatorname{dom}\left(\int_Y f d\mu\right)$.*

Now we specialize to the special case where the measure space is the spectrum of some unbounded operator A. In this case, we would like to know that such a projection-valued measure exists. And using this measure, we want to define functions of that operator.

Theorem 4.15. *Let A be a self-adjoint operator on $\mathcal{H}$ with domain $\operatorname{dom}(A)$. Then there exists a unique projection-valued measure μ^A on $\sigma(A)$ with values in $\mathcal{B}(\mathcal{H})$ such that*

$$\int_{\sigma(A)} \lambda d\mu^A(\lambda) = A. \tag{101}$$

Definition 4.16. *Let A be a self-adjoint operator on $\mathcal{H}$ and let $f : \sigma(A) \to \mathbb{R}$ be a measurable function. Define*

$$f(A) := \int_{\sigma(A)} f(\lambda) d\mu^A(\lambda). \tag{102}$$

I can't resist to say something personal at this point. For the longest time, in my physics classes, I've been told that one can define $f(A)$ for "reasonable" functions f. The only ones that seemed sense to me at the time were analytic functions, and I felt this was too restrictive. The above result generalizes this to almost any imaginable function.

Theorem 4.17. *Let A be a self-adjoint operator on $\mathcal{H}$ with domain $\operatorname{dom}(A)$. Then there exists a σ-finite measure μ on $\sigma(A)$, a direct integral $\int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda)$, and a unitary map $U : \mathcal{H} \to \int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda)$ such that*

$$[UAU^{-1}(s)](\lambda) = \lambda s(\lambda) \tag{103}$$

for all sections $s \in U(\mathrm{dom}(A))$, *where*

$$U(\mathrm{dom}(A)) = \left\{ s \in \int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda) \mid \int_{\sigma(A)} \|\lambda s(\lambda)\|_\lambda^2 d\mu(\lambda) < \infty \right\}. \tag{104}$$

If one is not happy with thinking about sections and one wants to remove the degeneracy from the description, there is also the following theorem that relates any function of an operator to a multiplication operator on *some* measurable space.

Theorem 4.18. *Let* A *be a self-adjoint operator on* $\mathcal{H}$ *with domain* $\mathrm{dom}(A)$. *Then there exists a* σ*-finite measure space* (Y, Σ, μ), *a measurable, real-valued function* $h : Y \to \mathbb{R}$, *and a unitary operator* $U : \mathcal{H} \to L^2(Y)$ *such that*

$$[UAU^{-1}(\psi)](\lambda) = h(\lambda)\psi(\lambda) \tag{105}$$

for all $\psi \in U(\mathrm{dom}(A))$ *and* $\lambda \in Y$, *where*

$$U(\mathrm{dom}(A)) = \left\{ \psi \in L^2(Y) \mid h\psi \in L^2(Y) \right\}. \tag{106}$$

4.4. **Stone's theorem.** All this business about bounded versus unbounded operators is actually quite difficult to keep track of. Is there any way to make our lives easier? You'd be surprised to hear that the answer to this question is "yes" and the reason is Stone's theorem. Stone's theorem basically says that the exponential of an unbounded operator is a *bounded* operator. Therefore, it is simpler to use the exponentiated versions of operators if one wants to avoid questions about domain and continuity. In this section, we will review Stone's theorem, describe its application to time evolution, and describe the Heisenberg algebra in terms of unitary operators via the Weyl relations.

Definition 4.19. *Let* $\mathcal{U}(\mathcal{H})$ *be the subspace of* $\mathcal{B}(\mathcal{H})$ *of unitary operators. A* one-parameter unitary group *on* $\mathcal{H}$ *is a function* $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ *satisfying*

$$U(0) = \mathrm{id}_{\mathcal{H}} \qquad \& \qquad U(s+t) = U(s)U(t) \tag{107}$$

for all $s, t \in \mathbb{R}$. *It is* strongly continuous *if*

$$\lim_{s \to t} \|U(t)\psi - U(s)\psi\| = 0 \tag{108}$$

for all $\psi \in \mathcal{H}$ *and all* $t \in \mathbb{R}$. *This amounts to saying that* $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ *is a continuous group homomorphism with respect to the strong operator topology on* $\mathcal{U}(\mathcal{H})$ *(and* not *the usual topology in terms of the operator norm).*

The fact that we need the strong operator topology is a bit annoying, however it turns out that *if* the infinitesimal generator (see the following definition) associated to U is a *bounded* operator, then U is continuous with respect to the *operator norm* topology (and vice versa).

Definition 4.20. *Let* U *be a strongly continuous one-parameter unitary group on* $\mathcal{H}$. *The* infinitesimal generator *of* U *is the operator* A *defined by*

$$\mathcal{H} \supset \mathrm{dom}(A) \ni \psi \mapsto -i \lim_{t \to 0} \left(\frac{U(t)\psi - \psi}{t} \right). \tag{109}$$

The domain is given by the set of ψ *for which the right-hand-side is defined.*

Given a *self-adjoint* operator A, one can use the exponential function to define the exponential and for every $t \in \mathbb{R}$ the operator

$$U(t) := e^{itA}. \tag{110}$$

Note that we need to use a self-adjoint operator because the spectral theorem requires us to do so.

Proposition 4.21. *A be a self-adjoint operator on $\mathcal{H}$ with domain $\mathrm{dom}(A)$. Define the operator $U(t)$ as above for every $t \in \mathbb{R}$. Then the following three things are true.*

(a) The function

$$\mathbb{R} \ni t \mapsto U(t) \tag{111}$$

is a strongly continuous one-parameter unitary group (in particular, it is a bounded operator on $\mathcal{H}$).

(b) For all $\psi \in \mathrm{dom}(A)$,

$$A\psi = -i \lim_{t \to 0} \left(\frac{U(t)\psi - \psi}{t} \right). \tag{112}$$

(c) For all $\psi \in \mathcal{H}$, if the limit

$$-i \lim_{t \to 0} \left(\frac{U(t)\psi - \psi}{t} \right) \tag{113}$$

exists, then $\psi \in \mathrm{dom}(A)$ and $A\psi = -i \lim_{t \to 0} \left(\frac{U(t)\psi - \psi}{t} \right)$.

The first statement is probably the most surprising. This says that one can take an *unbounded* self-adjoint operator and turn it into path of *bounded* unitary operators via the exponential. The second and third statement together say that

$$A = -i \lim_{t \to 0} \left(\frac{U(t) - I}{t} \right). \tag{114}$$

Stone's theorem is a sort of converse to the previous proposition.

Theorem 4.22 (Stone)**.** *Let U be a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then the infinitesimal generator A of U is a self-adjoint operator (in particular, it is densely defined) and $U(t) = e^{itA}$ for all $t \in \mathbb{R}$.*

time evolution
Weyl relation

5. Fifth lecture: degeneracy and perturbation theory—a geometric picture with several examples

In this lecture we will discuss the relationship between direct integrals, which were used in the spectral theorem in the last lecture, and degeneracy in quantum mechanics. We will then discuss perturbation theory and how this affects the degeneracy.

Example: two-dimensional harmonic oscillator, hydrogen atom, the quantum Hall effect

6. Sixth lecture: the Stone-von Neumann theorem

7. Seventh lecture:

References

[1] John C. Baez. *Quantum Quandaries: A Category-Theoretic Perspective*, in Structural Foundations of Quantum Gravity, Oxford University Press.
[2] Max Born. *Experiment and Theory in Physics*, Cambridge at the University Press 1944.
[3] Robert Eisberg and Robert Resnick. *Quantum Physics of Aroms, Molecules, Solids, Nuclei, and Particles*, Second Edition, John Wiley & Sons, 1974.
[4] Dan Freed and Constantin Teleman. *Relative Quantum Field Theory.*
[5] Carolyn Gordon and David Webb. *You Can't Hear the Shape of a Drum.* In American Scientist, Volume 84, 1996.
[6] Brian C. Hall. *Quantum Theory for Mathematicians*, Springer, 2013.
[7] Michael Kumarasen. *The Fourier Transform and Applications*—a talk on April 25, 2014 at the Mathematical Physics and Harmonic Analysis seminar at the CUNY Graduate Center.
[8] Leon A. Takhtajan. *Quantum Mechanics for Mathematicians*, the American Mathematical Society, 2008.
[9] Ultraviolet catastrophe - Wikipedia, the free encyclopedia. `http://en.wikipedia.org/wiki/Ultraviolet_catastrophe`