# Eugenio Hernández on "The Zak Transform and Some of its Mathematical Uses"

Notes by Arthur Parzygnat

October 10, 2014

The Zak transform is defined for functions $f : \mathbb{R} \to \mathbb{C}$ by

$$(Zf)(x,\xi) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i k\xi} \tag{1}$$

where $\xi \in \mathbb{R}$. It has many properties. First,

$$(Zf)(x,\xi+1) = (Zf)(x,\xi). \tag{2}$$

Therefore it suffices to consider $\xi \in \mathbb{R}/\mathbb{Z}$. Second,

$$(Zf)(x+1,\xi) = e^{2\pi i\xi}(Zf)(x,\xi) \tag{3}$$

by a change of variables. So it's not quite periodic in the first variable but the function is still determined by the values of $x \in \mathbb{R}/\mathbb{Z}$. Therefore, although it is not periodic, it is sufficient to define the Zach transform in $[0,1)^2$. $Z$ is well-defined for $f \in L^1(\mathbb{R})$. We need the series to converge absolutely, which means that

$$\int_0^1 \sum_{k\in\mathbb{Z}} |f(x+k)|dx = \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(y)|dy = \int_{\mathbb{R}} |f| < \infty. \tag{4}$$

Recall, given a function $f$, the Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx \tag{5}$$

on $\mathbb{R}$. Parseval's theorem says that if $f \in L^2(\mathbb{R})$, then

$$\|\mathcal{F}f\|_2 = \|f\|_2. \tag{6}$$

We also have Poisson's summation formula, which says that

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) = \sum_{k\in\mathbb{Z}} f(k). \tag{7}$$

Finally, we have another formula called the Shannon-Whittaker sampling theorem, which is the basis of digital technology. It says that if $\mathrm{supp}(\hat{f}) \subset [-B/2, B/2]$ (bump-limited), then

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{B}\right) \frac{\sin(\pi(Bx-k))}{\pi(Bx-k)} \tag{8}$$

where we sometimes call the latter function

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}. \tag{9}$$

This is called the sampling theorem because we sample the function at some fixed values. The number you need for a decent approximation is related to $B$. This is the foundations of DCT, digital communications technology (and also discrete cosine transform).

Our goal for this talk is to prove these three theorems using the Zak transform. The reason for this is because typically these theorems are non-trivial to prove. The Zak transform offers a simple way to prove them.

First recall that $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a basis of $L^2([0,1])$.

**Proposition 1.** *The Zak transform is an isometric isomorphism from $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$.*

*Proof.* It suffices to show that orthonormal bases go to orthonormal bases. The orthonormal basis for $L^2(\mathbb{R})$ that we will choose is rather clever and is given by

$$\{E_{n,k}(x) := e^{2\pi inx}\chi_{[0,1)}(x-k)\}_{n\in\mathbb{Z},k\in\mathbb{Z}}, \tag{10}$$

where $\chi_E$ is the indicator function on $E \subset \mathbb{R}$ defined by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}. \tag{11}$$

Then

$$\begin{aligned}
(ZE_{n,k})(x,\xi) &= \sum_{j\in\mathbb{Z}} e^{2\pi in(x+j)}\chi_{[0,1)}(x+j-k)e^{-2\pi ij\xi} && (12)\\
&= e^{2\pi inx}\sum_{l\in\mathbb{Z}} \chi_{[0,1)}(x+l)e^{-2\pi i(k+l)\xi} && (13)\\
&= e^{2\pi inx}e^{-2\pi ik\xi}\sum_{l\in\mathbb{Z}} \chi_{[0,1)}(x+l)e^{-2\pi il\xi} && (14)\\
&= e^{2\pi i(nx-k\xi)}\chi_{[0,1)}(x) && (15)\\
&= e^{2\pi i(nx-k\xi)} && (16)
\end{aligned}$$

which is a basis of $L^2([0,1))$. □

A bit of history. Joshua Zak published his ideas in 1967 and 1968. His second version is more detailed. He called it the $kq$-representation. He used it to describe electrons moving in a crystal.

**Definition 1.** A modified version of the Zak transform is given by

$$(\tilde{Z}g)(x,\xi) := (Zg)(\xi,-x) = \sum_{k\in\mathbb{Z}} g(\xi+k)e^{2\pi ikx}. \tag{17}$$

This is also an isometry from $L^2(\mathbb{R})$ to $L^2([0,1)^2)$. We also define $U : L^2([0,1)^2) \to L^2([0,1)^2)$ by

$$(Uf)(x,\xi) = e^{-2\pi ix\xi}f(x,\xi). \tag{18}$$

This is an isometry from $L^2([0,1)^2)$ to itself.

**Theorem 1.** *The following are true using our notation from above.*

*(a) First,*

$$(\tilde{Z}^{-1}UZ)(f) = \mathcal{F}f. \tag{19}$$

*(b) Second,*

$$(Z^{-1}U^*\tilde{Z})(g) = \mathcal{F}^{-1}g \tag{20}$$

*where*

$$(\mathcal{F}^{-1}g)(\xi) = \int_{\mathbb{R}} g(x)e^{2\pi ix\xi}dx \tag{21}$$

This is a result from a paper done jointly with Sikic, Weiss, and Wilson in 2010.

*Proof.* The inverse of the Zak transform is

$$(Z^{-1}\varphi)(x) = \int_0^1 \varphi(x,\xi)d\xi \tag{22}$$

and the inverse of the modified version of the Zak transform is

$$(\tilde{Z}^{-1}\varphi)(\xi) = \int_0^1 \varphi(x,\xi)dx. \tag{23}$$

Let's check the first is an inverse of the Zak transform

$$\begin{aligned}
((Z^{-1}(Zf))(x) &= \int_0^1 (Zf)(x,\xi)d\xi && (24)\\
&= \int_0^1 \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi ik\xi}d\xi && (25)\\
&= \sum_{k\in\mathbb{Z}} f(x+k)\int_0^1 e^{-2\pi ik\xi}d\xi && (26)\\
&= \sum_{k\in\mathbb{Z}} f(x+k)\delta_{k,0} && (27)\\
&= f(x). && (28)
\end{aligned}$$

It works! Thus, the second part of the theorem trivially follows by the following computation

$$\begin{aligned}
(Z^{-1}U^*\tilde{Z}g)(x) &= \int_0^1 (U^*\tilde{Z}g)(x,\xi)d\xi && (29)\\
&= \int_0^1 e^{2\pi ix\xi}(\tilde{Z}g)(x,\xi)d\xi && (30)\\
&= \int_0^1 \sum_{k\in\mathbb{Z}} g(\xi+k)e^{2\pi ix(k+\xi)}d\xi && (31)\\
&= \sum_{k\in\mathbb{Z}} \int_k^{k+1} g(\xi)e^{2\pi ix\xi}d\xi && (32)\\
&= \int_{\mathbb{R}} g(\xi)e^{2\pi i\xi x}d\xi && (33)\\
&= (\mathcal{F}^{-1}g)(x). && (34)
\end{aligned}$$

The rest of the proof is left to the reader. $\square$

Parseval's equality immediately follows from part (a) of the above theorem because each of the operators $\tilde{Z}^{-1}, U$, and $Z$ is an isometry.

Now we come to the Poisson summation formula. Let's write out (a) in the following way

$$(U(Z(f)))(x,\xi) = e^{-2\pi i x\xi} \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i k\xi} = \sum_{k\in\mathbb{Z}} (\mathcal{F}f)(\xi+k)e^{2\pi i k x}. \tag{35}$$

If you take $x = 0 = \xi$ then

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k), \tag{36}$$

which is exactly Poisson's summation formula.

Now let's prove the Shannon-Whittaker formula. Using the same expression, just take $x = 0$. Then we have

$$\sum_{k\in\mathbb{Z}} f(k)e^{-2\pi i k\xi} = \sum_{k\in\mathbb{Z}} (\mathcal{F}f)(\xi+k). \tag{37}$$

Suppose first that $\mathrm{supp}\hat{f} \subset [-1/2, 1/2]$. Then, for $\xi \in [-1/2, 1/2]$, we have

$$\sum_{k\in\mathbb{Z}} (\mathcal{F}f)(\xi+k) = (\mathcal{F}f)(\xi). \tag{38}$$

Therefore, we have

$$\sum_{k\in\mathbb{Z}} f(k)e^{-2\pi i k\xi} = (\mathcal{F}f)(\xi). \tag{39}$$

Then we have

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) \tag{40}$$

$$= \int_{-1/2}^{1/2} \hat{f}(\xi)e^{2\pi i x\xi} d\xi \tag{41}$$

$$= \int_{-1/2}^{1/2} \sum_{k\in\mathbb{Z}} f(k)e^{-2\pi i k\xi}e^{2\pi i x\xi} d\xi \tag{42}$$

$$= \sum_{k\in\mathbb{Z}} f(k) \int_{-1/2}^{1/2} e^{-2\pi i(k-x)\xi} d\xi \tag{43}$$

By a simple reparametrization, we can obtain the Shannon-Whittaker formula.

Another historical note. Andrew Weil also published a paper with the same mathematical transform. However, we name it after Zak. In 1950, Israel Gelfand gave similar proofs to what we have given here. Some other comments. You can write a Wigner distribution using the Zak transform. There is a paper by Janssen that has the formulas.